OUTPUT FEEDBACK STABILIZATION OF LINEAR SYSTEMS

By

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To My Mother

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In this dissertation, the output feedback stabilization
problem is studied from a more general framework. For any
$\alpha < \beta$, let S_α^n and $C_{\alpha,\beta}^n$ denote $\{(a_1, a_2, \ldots, a_n) \in R^n \mid x^n + \Sigma_{j=1}^n a_j x^{n-j} = 0$ implies $\text{Re}(x) < \alpha\}$ and $\{(a_1, a_2, \ldots, a_n) \in R^n \mid x^n + \sum_{j=1}^n a_j x^{n-j} = 0$ implies $|x - (\alpha+\beta)/2| < (\beta-\alpha)/2\}$ respectively. In chapter one, we illustrate that under the general
setting, the output feedback stabilization problem of a
system with single control can be converted to a problem of
examining whether a given affine set (depending on the
system) intersects S_0^n. Main results of this dissertation are
also mentioned. In chapter two, we obtain easy necessary
and sufficient conditions for a hyperplane H to intersect
S_α^n. We also show that $H \cap S_\alpha^n \neq \phi$ if and only if $H \cap RS_\alpha^n \neq \phi$
where $RS_\alpha^n = \{(a_1, a_2, \ldots, a_n) \in R^n \mid$ all the roots of $x^n + \Sigma_{j=1}^n a_j x^{n-j} = 0$ are real, distinct and strictly less than $\alpha\}$.
The results are based on the following property: "The convex
hulls of RS_0^n and S_0^n are both equal to the positive orthant

of R^n." In chapter three, we characterize hyperplanes which intersect $C_{\alpha,\beta}^n$ for arbitrary fixed $\alpha < \beta$. We also show that $co(C_{\alpha,\beta}^n)$ is an open simplex in R^n generated by some special vertices. In chapter four, we investigate the output feedback stabilization problem of a special delayed system. Chapter five contains a sufficient condition for a straight line to intersect S_α^n and a computer aided algorithm for determining the S_α-stabilizability of a given straight line. Some discussion of the directions for future research is included.

CHAPTER I
INTRODUCTION

Consider an unstable continuous linear time invariant control system:

$$(C) \quad \dot{x} = Fx + Gu, \quad y = Hx$$

where $F \in R^{n \times n}$, $G \in R^{n \times m}$, $H \in R^{r \times n}$. One of the problems which have been receiving a great deal of attention in control theory is the Output Feedback Stabilization problem. It is to determine the existence of an output feedback $u = Ky$ or even to find such $K \in R^{m \times r}$ so that the closed loop system becomes stable. In other words, it is to find a matrix $K \in R^{m \times r}$ such that all the roots of the following equation have negative real parts.

$$\det(xI - I - GKH) = 0$$

Wonham [1] had proved the following theorem.

Theorem 1.1.1. For a control system defined as in (C), let $H = I$. Then the pair (F,G) is controllable, namely, rank $(G, FG, \ldots, F^{n-1}G) = n$ (see Kalman et al. [11]) if and only if for every choice of $(a_1, a_2, \ldots, a_n) \in R^n$, there is a matrix $K \in R^{m \times r}$ such that $\det(xI - F - GKH) = x^n + \sum_{j=1}^{n} a_j x^{n-j}$. []
Note that in this theorem, physically, the assumption $H = I$ implies that all the states are measurable. This is unlikely to most practical situations.

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A considerable amount of research in the literature has been devoted to finding a controller designed from the available state measurements to stabilize the system. One approach is to obtain estimates of the inaccessible states from a Luenberger Observer (Luenberger [1], [2], [3]) and use these estimates along with the available state measurements to construct a feedback control. Along the same line, Brasch and Pearson [1] showed that arbitrary pole placement can be obtained by adding a compensator to the system. Both are powerful methods for stabilizing a system; yet, one pays the price of increasing the dimension of the system. Hence, it is still desirable to know whether a constant gain feedback can be designed directly from the available state measurements. Most of the efforts of the researchers such as Miller, Cochran and Howze [1], Luus [1], McBrinn and Roy [1], and Sirisena and Choi [1] were devoted to the study of numerical schemes for finding the feedback gain matrix k. In a paper by Anderson, Bose and Jury [1], the problem of output feedback stabilization was related to the decidability problems (see Jacobson [1]) and these authors showed how the decision methods of Tarski [1] and Seidenberg [1] can be applied to solve the stabilization problem.

In this dissertation, we shall look at the output feedback stabilization problem from a more general point of view described in the following.

Definition 1: Given a set K, an n-dimensional linear output feedback system with parameters in K is defined to be an ordered pair $(\pi, K)^n$, where π is a mapping from K into $R^{n \times n} \times R^{n \times m} \times R^{m \times n}$ for some fixed m, which associates with each parameter $K \in K$ a triple $(A(K), B(K), C(K))$ with $A(K) \in R^{n \times m}$, $B(K) \in R^{n \times m}$ and $C(K) \in R^{m \times n}$. For brevity, in the sequel we simply say "a system $(\pi, K)^n$ with parameters in K."

For a fixed m and n, let χ be a mapping from $R^{n \times n} \times R^{n \times m} \times R^{m \times n}$ into R^n which assigns each (A, B, C) a vector $(a_1, a_2, \ldots, a_n) \in R^n$ where a_j's satisfy $x^n + \Sigma_{j=1}^n a_j x^{n-j} = \det(xI - A - BC)$. Let

$$S_\alpha^n = \{(a_1, a_2, \ldots, a_n) \in R^n \mid x^n + \Sigma_{j=1}^n a_j x^{n-j} = 0 \text{ implies}$$
$$Re(x) < \alpha\}$$

$$RS_\alpha^n = \{(a_1, a_2, \ldots, a_n) \in R^n \mid \text{all the roots of } x^n + \Sigma_{j=1}^n a_j x^{n-j} = 0$$
$$\text{are real, distinct and strictly less than } \alpha\}$$

Definition 2: For a fixed $\alpha \in R$, an n-dimensional linear output feedback system $(\pi, K)^n$ with parameters in K, is said to be S_α-stabilizable if $\chi \pi(K) \cap S_\alpha^n \neq \phi$. A system $(\pi, K)^n$ with parameters in K is said to be RS_α^n-stabilizable if $\chi \pi(K) \cap RS_\alpha^n \neq \phi$.

If one lets $K = R^{m \times r}$ and defines $\pi(K) = (F, G, KH)$ for each $K \in K$ where F, G, H are the matrices given in (C), then $\chi \pi(K)$ is equal to the coefficients of the characteristic polynomial of (C) with output feedback $u = Ky$. Note that (C) is stabilizable if and only if $\chi \pi(K) \cap S_0^n \neq \phi$, or equivalently, by definition, $(\pi, K)^n$ is S_0-stabilizable. Suppose $H = I$ and (F, G) forms a completely controllable pair. Under the same π

as above, we observe that the well known result (theorem 1.1.1) of Wonham [1] means then $\chi\pi(K) = R^n$.

For any system (C) with single control i.e., $m = 1$, it was shown that (see Byrnes [1], Kamen [1])

$$\det(xI-F-GKH) = \det(xI-F) + kH \, \text{adj}(xI-F)G \qquad (1.1)$$

With $K = R^{1 \times r}$ and the same π as above, the relation (1.1) implies that $\chi\pi(K) = \chi\pi(0) + KM$ where M is a $r \times n$ matrix depending on F,G,H. Hence, $\chi\pi(K)$ forms an affine set in R^n.

Now consider the class of all systems $(\pi,K)^n$ for which $\chi\pi(K)$ forms an affine set. This is a large class which includes all systems (C) with single control as the discussion in the above shows. Observe that for a fixed $\alpha \in R$ if one can characterize affine sets which intersect S^n_α, then S^n_α-stabilizability of a system $(\pi,K)^n$ for which $\pi(K)$ forms an affine set is characterized. In particular, the problem of output feedback stabilization for systems with single control is resolved. As a first step in this direction we consider mainly, the case of hyperplanes in this dissertation.

Chapter 2 is devoted to the study of characterizing hyperplanes which intersect S^n_α and RS^n_α. We prove that a hyperplane $H = \{(a_1,a_2,\ldots,a_n) \in R^n \mid A_0 + \Sigma^n_{j=1}A_j a_j = 0\}$ intersects S^n_0 if and only if $H \cap R^n_+$ where R^n_+ is the positive orthant of R^n, i.e., $\{(a_1,a_2,\ldots,a_n) \in R^n \mid a_j > 0\}$. Furthermore, it is shown that $H \cap S^n_0 \neq \phi$ is equivalent to $H \cap RS^n_0 \neq \phi$. We obtain also the criterion which states $H \cap S^n_0 \neq \phi$ if and only if $A_i A_j < 0$

for some $i \neq j$, $i,j = 0,1,2,\ldots,n$. The simplicity of these properties results from the following lemma:

"The convex hulls of S_0^n and RS_0^n are equal to the positive orthant R_+^n of R^n i.e., $co(S_0^n) = co(RS_0^n) = R_+^n$."

The author would like to acknowledge that for the case of $n = 4$, the lemma was first shown by Professor Popov. Based on the idea from that proof, this author extended the lemma for arbitrary n. Also, the proof for general n, was greatly simplified by a suggestion from Professor Popov.

The results mentioned above are generalized to characterize all hyperplanes which intersect S_α^n or RS_α^n for an arbitrary fixed $\alpha \in R$. Observe that for any system $(\pi, K)^n$ with parameters in K, there exists always $\alpha \in R$ such that $\chi\pi(K) \cap S_\alpha^n \neq \phi$. From the point of view of stability, it is desirable to have that α "as negative as possible" to reduce the transient time. However, if $\chi\pi(K)$ forms only a hyperplane in R^n, then one can not expect the relation $\chi\pi(K) \cap S_\alpha^n \neq \phi$ to hold for any $\alpha \in R$. We show that given a hyperplane H the value $\inf\{\alpha \mid H \cap S_\alpha^n \neq \phi\}$ is equal to the negative of the maximal real root of a specially constructed polynomial. Some computation aspects such as how to find a $(a_1, a_2, \ldots, a_n) \in H \cap S_\alpha^n$, if $H \cap S_\alpha^n \neq \phi$, are discussed. An example chosen from Anderson et al. [1] is included.

Given a discrete linear time invariant control system

(S) $x_{k+1} = Fx_k + Gu$, $y_k = Hx_k$,

(S) is stabilizable with output feedback if and only if there exist $K \in R^{m \times r}$ such that the modulus of all the roots of $\det(xI - F - GKH) = 0$ are less than one. Given $\alpha < \beta$, let

$$C^n_{\alpha,\beta} = \{(a_1, a_2, \ldots, a_n) \in R^n \mid x^n + \Sigma^n_{j=1} a_j x^{n-j} = 0 \text{ implies}$$
$$|x - (\beta+\alpha)/2| < (\beta-\alpha)/2\}$$

$$RC^n_{\alpha,\beta} = \{(a_1, a_2, \ldots, a_n) \in R^n \mid \text{all the roots of } x^n + \Sigma^n_{j=1} a_j x^{n-j} = 0$$

are real, distinct and strictly between α and $\beta\}$.

By taking $K = R^{m \times r}$ and $\pi(K) = (F, G, KH)$ for each $K \in K$, one can regard (S) with output feedback as a system $(\pi, K)^n$ with parameters in K. (S) is stabilizable by an output feedback if and only if $\chi\pi(K) \cap C^n_{-1,1} \neq \phi$. The main objective of chapter 3 is to find a necessary and sufficient condition for a hyperplane to have nonempty intersection with $C^n_{-1,1}$. For the reason of mathematical interest, we obtain necessary and sufficient conditions for a hyperplane to intersect $C^n_{\alpha,\beta}$ for general $\alpha < \beta$. It is interesting to point out that given $\alpha < \beta$, the $C^n_{\alpha,\beta}$ and $RC^n_{\alpha,\beta}$ are both equal to the open n simplex in R^n generated by the n+1 vertices $(q_{i1}, q_{i2}, \ldots, q_{in}) \in R^n$ $i = 0,1,\ldots,n$ where $x^n + \Sigma^n_{j=1} q_{ij} x^{n-j} = (x-\alpha)^{n-i} (x-\beta)^i$.

In chapter 4, we consider the stabilization of a special kind of system with delay:

(D) $\quad \dot{x}(t) = \tilde{a}x(t) + \tilde{b}x(t-1) + u, \quad y = \tilde{c}x(t) + \tilde{d}x(t-1), \quad u = ky$

With $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ fixed, one wishes to find $k \in R$ such that all the roots of $\lambda + (\tilde{a}+k\tilde{c}) + (\tilde{b}+k\tilde{d})e^{-\lambda} = 0$ have negative real parts. Given $\alpha \in R$, let

$$\mathcal{D}_\alpha = \{(a,b) \in R^n \mid \lambda + a + be^{-\lambda} = 0 \text{ implies } Re\lambda < \alpha\}$$

This stabilization problem turns out to be a problem of determining whether a given straight line intersects $\mathcal{D}_0$. We obtain a necessary and sufficient condition for a straight line to intersect $\mathcal{D}_\alpha$ for general $\alpha \in R$. It is interesting to note that for any given hyperplane H, the number $\inf\{\alpha \mid H \cap S_\alpha^n \neq \phi\}$ is always bounded from below (theorem 2.3.1). However, in this delay case, there are lines $H \subset R^2$ (hyperplane in R^2) for which the number $\inf\{\alpha \mid H \cap \mathcal{D}_\alpha \neq \phi\} = -\infty$ (theorem 4.2.4).

One of the problems concerning the stabilization of systems with delay is that "Can one stabilize a system with delay by a feedback without delay?" In the end of the chapter 4 we show with an example, that this can not be done in general, despite the fact that this can be done for $(\mathcal{D})$.

Despite all the easy criteria for determining different kinds of stabilizabilities obtained for hyperplanes in chapters 2, 3 and 4, we are unable to find an easy necessary and sufficient condition for affine sets with dimension strictly lower than n-1 to intersect S_α^n. As an attempt, in chapter 5, we obtain a sufficient condition for a straight line to intersect S_α^n. Also, a computer aided algorithm is proposed for determining the S_α-stabilizability of a straight line. Some directions for future research are discussed.

S_α^n-STABILIZABLE AND RS_α^n-STABILIZABLE HYPERPLANES

For a fixed n, let $(\pi, K)^n$ be a n dimensional system with parameters in K. Recall that K is a set of parameters and π is a mapping from K into $R^{n \times n} \times R^{n \times m} \times R^{m \times n}$ for some fixed m, which associates each parameter $K \in K$, a triple denoted by (A(K), B(K), C(K)). Let χ be the mapping from $R^{n \times n} \times R^{n \times m} \times R^{m \times n}$ into R^n which assigns each $(A, B, C) \in R^{n \times n} \times R^{n \times m} \times R^{m \times n}$ a vector $(a_1, a_2, \ldots, a_n) \in R^n$ where $x^n + \sum_{j=1}^{n} a_j x^{n-j} = \det(xI - A - BC)$.

The main purpose of this chapter is to characterize the S_α^n-stabilizability and RS_α^n-stabilizability of systems $(\pi, K)^n$ for which $\chi\pi(K)$ forms a hyperplane in R^n. In other words, one wishes to characterize hyperplanes which intersect S_α^n and RS_α^n.

Recall that for a given $\alpha \in R$,

$$S_\alpha^n = \{(a_1, a_2, \ldots, a_n) \in R^n \mid x^n + \sum_{j=1}^{n} a_j x^{n-j} = 0 \text{ implies } Re(x) < \alpha\}$$

$$RS_\alpha^n = \{(a_1, a_2, \ldots, a_n) \in R^n \mid \text{ all the roots of } x^n + \sum_{j=1}^{n} a_j x^{n-j} = 0$$

are real, distinct and strictly less than $\alpha\}$.

Definition: Given $\alpha \in R$, a hyperplane $H \subset R^n$ is said to be S_α^n-stabilizable if $H \cap S_\alpha^n \neq \phi$. A hyperplane H is said to be RS_α^n-stabilizable if $H \cap RS_\alpha^n \neq \phi$.

The technique which we use in here is based on a simple property that a hyperplane in R^n intersects an open connected set in R^n if and only if the hyperplane intersects the convex hull of that open connected set. In lemma 2.1.5, we prove that $co(RS_0^n) = co(S_0^n) = R_+^n$. By an invertible affine transformation T_α and lemma 2.1.5, we obtain the descriptions of $co(S_\alpha^n)$ and $co(RS_\alpha^n)$. Section 2 contains some criteria for determining the S_α^n-stabilizability of a hyperplane. We show that the S_α^n-stabilizability and RS_α^n-stabilizability are equivalent for hyperplanes. In particular, we find that a hyperplane is S_0^n-stabilizable if and only if the hyperplane intersects the positive orthant. Finally, in section 3, the value $\inf\{\alpha | H \cap S_\alpha^n \neq \phi\}$ for a given hyperplane $H \subset R^n$ is computed and some consideration of actually finding a point $(a_1, a_2, \ldots, a_n) \in H \cap S_\alpha^n \neq \phi$ is included.

2.1   Convex Hulls of S_α^n and RS_α^n.

Let us begin with some basic properties of S_α^n and RS_α^n which will be needed later. Recall that $S_\alpha^n = \{\Sigma_{i=1}^n a_i \vec{e}_i \in R^n |$ $x^n + \Sigma_{j=1}^n a_j x^{n-j}$ implies $Re(x) < \alpha\}$ and $RS_\alpha^n = \{\Sigma_{j=1}^n a_j \vec{e}_j \in R^n |$ all the roots of x of $x^n + \Sigma_{j=1}^n a_j x^{n-j} = 0$ are real, distinct and $x < \alpha\}$.

<u>Lemma 2.1.1.</u>   (i) RS_α^n and S_α^n are open connected sets in R^n for any $\alpha \in R$;

(ii) $S_\alpha^n \subset S_\beta^n$ if and only if $\alpha < \beta$;

(iii) $RS_\alpha^n \subset RS_\beta^n$ if and only if $\alpha < \beta$;

(iv) $RS_\alpha^n \subset S_\alpha^n \subset R_+^n$ for every $\alpha < 0$.

The proof of (i) will be presented in appendix I

the others follow straightforwardly from the definitions of

S_α^n and RS_α^n together with the fact that $S_0^n \subset R_+^n$.

Since Hermite [1], enormous research had been devoted to

the study of the set S_0^n. Algebraically, S_0^n can be characterized

by the well known criteria of Routh and Hurwitz or Liénard

and Chipart (Gantmacher [1]), in terms of system of nonlinear

inequalities. From these criteria, it follows that S_0^n is

subset of R_+^n. Since the roots of a polynomial depend contin-

uously on the coefficients, the boundary of S_0^n consists of

coefficients of polynomials which have either a zero root or

a pair of purely imaginary roots. Indeed, the D-decomposition

method of Yu Naimark says that the hypersurfaces $H_1 = \{(a_1, a_2,$

$\ldots, a_n) \in R^n \mid a_n = 0\}$ and $H_2 = \{(a_1, a_2, \ldots, a_n) \in R^n \mid x^n +$

$\Sigma_{j=1}^n a_j x^{n-j} = 0$ has a pair of purely imaginary roots} divide

the coefficient space R^n into n+1 disjoint regions; and the

polynomials with coefficients chosen from the same region will

have the same number of roots with positive real parts. The

example below illustrates the case n = 2.

Example: For n = 1, $S_0^2 = \{(a_1, a_2) \mid a_1 > 0, a_2 > 0\}$ and

$RS_0^2 = \{(a_1, a_2) \mid a_1 > 0, a_2 > 0, a_1^2 > 4a_2\}$. On the curve

$a_1^2 = 4a_2$, the polynomials will have identical roots. In

figure 2.1, p represents the number of roots with positive

real parts.

The sets S_α^n, for $\alpha \in R$, as exemplified by S_0^n, are deter-

mined by collections of nonlinear inequalities for a_i's. In
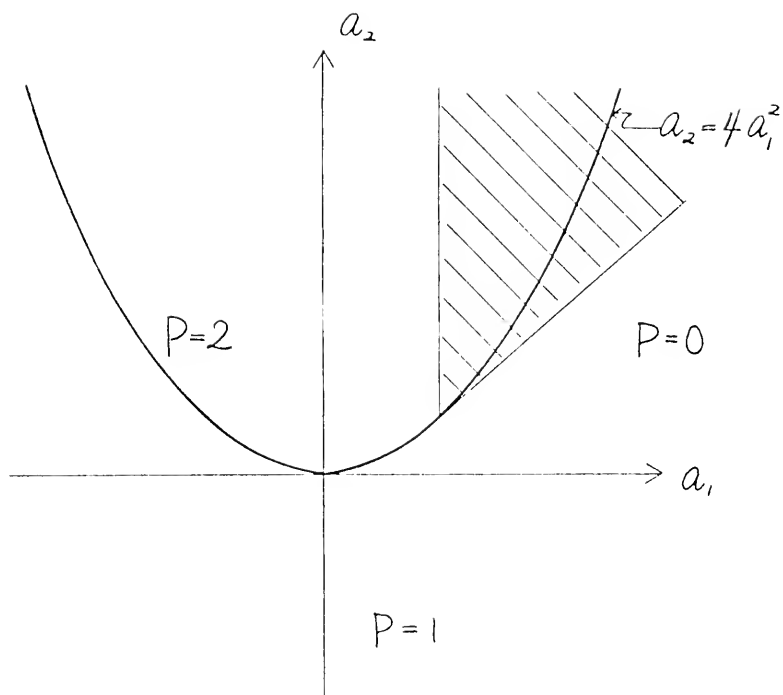
a_2

$a_2 = 4a_1^2$

P=2

P=0

a_1

P=1

Figure 2.1

general, for a given hyperplane $H \subset R^n$, it is not easy to determine if H is S_α-stabilizable by considering directly the intersection of H with S_α^n. However, because S_α^n is open connected in R^n, the following lemma shows that one can consider instead the intersection of H with $co(S_\alpha^n)$, a set which is larger and more regular than S_α^n itself.

Lemma 2.1.2. Let H be a hyperplane in R^n, and let $S \subset R^n$ be an open connected set. Then $H \cap S \neq \phi$ if and only if $H \cap co(S) \neq \phi$.

Proof: ($\Rightarrow$). Trivial, since $S \subseteq co(S)$.

($\Leftarrow$). Let E_1 and E_2 be the open half spaces corresponding to the hyperplane H. Assume the contrary that $H \cap S = \phi$. Then, by the hypothesis that S is open and connected, we have either $S \subseteq E$, or $S \subseteq E_2$. Without loss of generality, let us assume $S \subseteq E_1$. Since E_1 is convex and $S \subseteq E_1$, it follows that $H \cap co(S) \subset H \cap co(E_1) = H \cap E_1 = \phi$. This contradicts the hypothesis $H \cap co(S) \neq \phi$. Therefore, one concludes that $H \cap S \neq \phi$. □

Using this lemma, the S_α^n-stabilizable hyperplanes can be characterized as those which intersect $co(S_\alpha^n)$. However, if one wishes to take the full advantage of this characterization of S_α^n-stabilizable hyperplanes, one must know what $co(S_\alpha^n)$ looks like. This will be the aim of the rest of this section. First, we shall prove the main lemma mentioned in the intro-duction (i.e., $co(S_0^n) = co(RS_0^n) = R_+^n$). Then, we shall use this lemma to obtain descriptions of $co(S_\alpha^n)$ and $co(RS_\alpha^n)$ for arbitrary $\alpha \in R$.

In order to prove the main lemma, since $co(S_0^n)$, $co(RS_0^n)$ and R_+^n are open convex sets, it suffices to show that their closures are equal i.e., $\overline{co(S_0^n)} = \overline{co(RS_0^n)} = \overline{R_+^n}$ (see e.g., Rockafellar [1], p. 45). This is what we establish in the following lemmas.

Lemma 2.1.3. The origin $\theta = (0,0,\ldots,0) = \Sigma_{j=1}^n 0\vec{e}_j$ is contained in $\overline{co(RS_0^n)}$.

Proof: Let δ be a positive number. Consider the following polynomial

$$\prod_{j=1}^n (x+j\delta) = x^n + \sum_{j=1}^n P_j(\delta) x^{n-j}$$

Observe that for each j, $P_j(\delta) = c_j \delta^j$ where c_j is a positive real number. Therefore, for any $\varepsilon > 0$, one can find $\tilde{\delta} > 0$ such that $P_j(\tilde{\delta}) < n^{-1/2}\varepsilon$ for every $j = 1,2,\ldots,n$. Note that $\prod_{j=1}^n (x+j\tilde{\delta}) = 0$ has only stable roots $x = -j\tilde{\delta}$, $j = 1,2,\ldots,n$. Thus, by the definition of RS_0^n,

$$\sum_{j=1}^n P_j(\tilde{\delta})\vec{e}_j \in RS_0^n \qquad (2.1)$$

Also, one has

$$\| \sum_{j=0}^n P_j(\tilde{\delta})\vec{e}_j - \sum_{j=1}^n 0\vec{e}_j \| = (\sum_{j=1}^n P_j^2(\tilde{\delta}))^{1/2} < (\sum_{j=1}^n \frac{\varepsilon^2}{n})^{1/2} = \varepsilon$$

By (2.1), it follows that $\Sigma_{j=1}^n P_j(\tilde{\delta})\vec{e}_j \in N_\varepsilon(0) \cap RS_0^n$. Hence, $0 \in \overline{RS_0^n} \subset \overline{co(RS_0^n)}$. $\qquad \|$

Let $\{\vec{e}_j\}_{j=1}^n$ be the standard basis of R^n.

Lemma 2.1.4. For any $r > 0$ and any positive integer k, the point $r\vec{e}_k$ is contained in $co(RS_0^n)$.

Proof: In this proof j will be used to represent positive integers. Take r and k as specified in the lemma.

For arbitrary $q > 0$ and $\delta_i > 0$, $i = 1,2,\ldots,n$ where $\delta_i \neq \delta_j$ if $i \neq j$, let $P_j(q,\delta_1,\delta_2,\ldots,\delta_n)$ be the polynomial of q and δ_i's (also written $P_j(q,\delta_i)$) obtained from the following polynomial expansion

$$\prod_{i=1}^{k} (x+q+\delta_i) \cdot \prod_{i=k+1}^{n} (r+\delta_i) = x^n + \sum_{j=1}^{n} P_j(q,\delta_1,\delta_2,\ldots,\delta_n)x^{n-j}$$

Note that for every $j \leq k$, the highest degree of variable q in $P_j(q,\delta_i)$ is j. In particular,

$$P_k(q,\delta_i) = q^k + \tilde{P}_k(q,\delta_i) \tag{2.2}$$

The degree of q of $\tilde{P}_k(q,\delta_i)$ is at most $k-1$. Fix $\delta_i = \tilde{\delta}_i > 0$, $i = 1,2,\ldots,n$. For every $\varepsilon > 0$, choose $q_0 > 0$ large enough such that for all j, $1 \leq j \leq k-1$

$$\frac{r}{q_0^k} P_j(q_0,\tilde{\delta}_1,\tilde{\delta}_2,\ldots,\tilde{\delta}_n) < \frac{\varepsilon}{\sqrt{n}} \tag{2.3}$$

and

$$\frac{r}{q_0^k} \tilde{P}_k(q_0,\tilde{\delta}_1,\tilde{\delta}_2,\ldots,\tilde{\delta}_n) < \frac{\varepsilon}{\sqrt{n}} \tag{2.4}$$

as well as

$$\max\{2,r\} < q_0^k \tag{2.5}$$

This is possible because the degrees of q in $P_j(q,\tilde{\delta}_i)$ are less than or equal to $k-1$ for all $j \leq k-1$ and the same property is true for $\tilde{P}_k(q,\tilde{\delta}_i)$. Since for any $j > k$, $P_j(q_0,0,0,\ldots,0) = 0$ and since $P_j(q_0,\delta_1,\delta_2,\ldots,\delta_n)$ are polynomials of δ_i's with positive coefficients, one can find $0 < \hat{\delta}_i < \tilde{\delta}_i$, $i = 1,2,\ldots,n$, where $\hat{\delta}_i \neq \hat{\delta}_j$ if $i \neq j$, such that for each $j > k$

$$\frac{r}{q_0^k} P_j(q_0,\hat{\delta}_1,\hat{\delta}_2,\ldots,\hat{\delta}_n) < \frac{\varepsilon}{\sqrt{n}} \tag{2.6}$$

Also, since for each i, $\hat{\delta}_i < \tilde{\delta}_i$, by the property that $P_j(q,\delta_i)$, $j = 1,2,\ldots,n$ have positive coefficients, one obtains for each $j \leq k-1$

$$\frac{r}{q_0^k} P_j(q_0,\hat{\delta}_1,\hat{\delta}_2,\ldots,\hat{\delta}_n) < \frac{r}{q_0^k} P_j(q_0,\tilde{\delta}_1,\tilde{\delta}_2,\ldots,\tilde{\delta}_n) < \frac{\varepsilon}{\sqrt{n}} \tag{2.7}$$

and similarly,

$$\frac{r}{q_0^k} \tilde{P}_k(q_0,\hat{\delta}_1,\hat{\delta}_2,\ldots,\hat{\delta}_n) < \frac{r}{q_0^k} \tilde{P}_k(q_0,\tilde{\delta}_1,\tilde{\delta}_2,\ldots,\tilde{\delta}_n) < \frac{\varepsilon}{\sqrt{n}} \tag{2.8}$$

Now, consider the following polynomial equation

$$\prod_{j=1}^{k} (x+q_0+\hat{\delta}_i) \prod_{j=k+1}^{n} (x+\hat{\delta}_i) = x^n + \sum_{j=1}^{n} P_j(q_0,\hat{\delta}_1,\hat{\delta}_2,\ldots,\hat{\delta}_n) x^{n-j}$$

It has only stable roots $x = -(q_0+\hat{\delta}_i) < 0$, $i = 1,2,\ldots,k$ and $x = -\hat{\delta}_i < 0$, $i = k+1,\ldots,n$. This, by the definition of RS_0^n, implies that the point

$$\vec{z}_\varepsilon = \sum_{j=1}^{n} P_j(q_0,\hat{\delta}_1,\hat{\delta}_2,\ldots,\hat{\delta}_n)\vec{e}_j \tag{2.9}$$

belongs to RS_0^n. Hence, using the relation $RS_0^n \subset \overline{co(RS_0^n)}$, we obtain

$$\vec{z}_\varepsilon \in \overline{co(RS_0^n)}$$

From Lemma 2.1.4, $\underline{\theta = \sum_{j=1}^n 0\vec{e}_j} \in \overline{co(RS_0^n)}$. Since $\overline{co(RS_0^n)}$ is convex, the set $\overline{co(RS_0^n)}$ also contains the line segment with the origin and $\vec{z}_\varepsilon$ as end points. In particular, $(r/q_0^k)\vec{z}_\varepsilon \in \overline{co(RS_0^n)}$, since $0 < r < q_0^k$. By (2.9) and (2.2), we have

$$\| r\vec{e}_k - \frac{r}{q_0^k} \vec{z}_\varepsilon \| = \{ \sum_{j=1}^{k-1} (\frac{r}{q_0^k} P_j(q_0, \hat{\delta}_i))^2 + (\frac{r}{q_0^k} \widetilde{P}(q_0, \hat{\delta}_i))^2$$

$$+ \sum_{j=k+1}^n (\frac{r}{q_0^k} P_j(q_0, \hat{\delta}_i))^2 \}^{1/2} \qquad (2.10)$$

We now prove that $\| r\vec{e}_k - (r/q_0^k)\vec{z}_\varepsilon \| < \varepsilon$. Let RHS denote the right hand side of (2.10). Then, by (2.6), (2.7), and (2.8) one has

$$\text{RHS} < \{ (k-1)\frac{\varepsilon^2}{n} + \frac{\varepsilon^2}{n} + (n-k)\frac{\varepsilon^2}{n} \}^{1/2} = \varepsilon$$

Therefore, we have shown that for every $\varepsilon > 0$, there exists a point $\frac{r}{q_0^k} \vec{z}_\varepsilon \in N_\varepsilon(r\vec{e}_k) \cap \overline{co(RS_0^n)}$. Hence, $r\vec{e}_k \in \overline{co(RS_0^n)}$. □

Now we state the main lemma.

<u>Lemma 2.1.5.</u> For every positive integer n, $\overline{co(RS_0^n)} = \overline{co(S_0^n)} = R_+^n$.

<u>Proof:</u> First, observe that $R_+^n = \{ \sum_{k=1}^n r_k(r\vec{e}_k) \mid r > 0, \sum_{k=1}^n r_k = 1, r_k > 0 \}$. Since, by lemma 2.1.4, $r\vec{e}_k \in \overline{co(RS_0^n)}$

for every $r > 0$ and every $k = 1, 2, \ldots, n$. Hence, it follows

that $R_+^n \subseteq \overline{co(co(RS_0^n))} = \overline{co(RS_0^n)}$. Conversely, by lemma 2.1.1

we have $\overline{co(RS_0^n)} \subseteq \overline{co(S_0^n)} \subseteq \overline{co(R_+^n)} = \overline{R_+^n}$. Therefore, $\overline{R_+^n} =$

$\overline{co(RS_0^n)} = \overline{co(S_0^n)}$. Since for any n dimensional convex set

$S \subseteq R^n$, interior $(\overline{S}) =$ interior (S) (see Rockafellar [1],

p. 46) also, since $co(RS_0^n)$, $co(S_0^n)$ and R_+^n are open, one con-

cludes that $co(RS_0^n) = co(S_0^n) = R_+^n$. $\quad\quad\quad\quad\quad\quad$ []

$\quad\quad$ Before we set out to find RS_α^n and S_α^n, let us mention the

following remark.

Remark: If T is an invertible affine transformation of R^n,

then for any $S \subseteq R^n$,

$$T(co(S)) = co(T(S))$$

This follows easily from the following facts:

$\quad\quad$(1) Every point $z \in co(S)$ can be represented as a con-

vex combination of points from S, i.e., $z = \Sigma_i k_i s_i$ with $s_i \in S$

and $k_i > 0$, $\Sigma_i k_i = 1$;

$\quad\quad$(2) $T(\Sigma_i k_i s_i) = \Sigma_i k_i T(s_i)$, since T is invertible affine.

$\quad\quad$Using this remark, one observes that if there exists an

invertible affine transformation T_α on R^n such that $T_\alpha(S_\alpha^n) = S_0^n$,

then

$$T_\alpha(co(S_\alpha^n)) = co(T_\alpha(S_\alpha^n)) = co(S_0^n) = R_+^n$$

Hence, the convex hull of S_α^n, namely, $co(S_\alpha^n)$ will simply be

the set of all points $(a_1, a_2, \ldots, a_n) \in R^n$ whose image under

T_α belongs to R_+^n.

Such a transformation does indeed exist, as we will show in the next lemma.

For any fixed $\alpha \in R$, let T_α be the transformation on R^n such that for every $(a_1, a_2, \ldots, a_n) \in R^n$,

$$T_\alpha(a_1, a_2, \ldots, a_n) = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$$

where $\tilde{a}_i$'s are obtained from the following polynomial expansion:

$$(x+\alpha)^n + \sum_{j=1}^{n} a_j(x+\alpha)^{n-j} = x^n + \sum_{j=1}^{n} \tilde{a}_j x^{n-j} \tag{2.11}$$

We remark that this transformation was considered in a paper by Hughes and Nguyen [1], and the explicit expression of $\tilde{a}_j$ was also given there. For completion we include the proof.

<u>Lemma 2.1.6.</u> T_α is an invertible affine transformation on R^n such that

$$T_\alpha(RS^n_\alpha) = RS^n_0 \quad \text{and} \quad T_\alpha(S^n_\alpha) = S^n_0$$

<u>Proof:</u>   (i) T_α is invertible and affine.

We first observe that, by Taylor's theorem, every polynomial $p(y) = y^n + \sum_{j=1}^{n} a_j y^{n-j}$ can be expressed as

$$p(y) = \sum_{j=0}^{n} \frac{p^{(j)}(\alpha)}{j!}(y-\alpha)^j \tag{2.12}$$

Thus, by substituting $y = x + \alpha$, (2.12) becomes

$$\sum_{j=0}^{n} a_j(x+\alpha)^{n-j} = p(x+\alpha) = \sum_{j=0}^{n} \frac{p^{(n-j)}(\alpha)}{(n-j)!} x^{n-j}$$

with $a_0 = 1$. In view of (2.11), it follows that, for each j, $1 \leq j \leq n$

$$\tilde{a}_j = \frac{p^{(n-j)}(\alpha)}{(n-j)!} = \sum_{i=0}^{j} \binom{n-i}{j-i} \alpha^{j-i} a_i \qquad (2.13)$$

To show that T_α is affine, let $(a_1', a_2', \ldots, a_n')$ and $(a_1'', a_2'', \ldots, a_n'')$ be arbitrary two points of R^n. Then, for each i, $1 \leq i \leq n$ and for $0 \leq \lambda \leq 1$,

$$\tilde{a}_j = \sum_{i=0}^{j} \binom{n-i}{j-i} \alpha^{j-i} [\lambda a_i' + (1-\lambda) a_i''] = \lambda [\sum_{i=0}^{j} \binom{n-i}{j-i} \alpha^{j-i} a_i'] +$$

$$+ (1-\lambda) [\sum_{i=0}^{j} \binom{n-i}{j-i} \alpha^{j-i} a_i'']$$

Hence, T_α is affine. Represented in matrix form,

$$T_\alpha(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n) Q_\alpha + (\binom{n}{1}\alpha, \binom{n}{2}\alpha^2, \ldots, \binom{n}{n}\alpha^n)$$

where $Q_\alpha = (q_{ij})$ is an $n \times n$ matrix with entries $q_{ij} = \binom{n-i}{j-i}\alpha^{j-i}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$. Observe that Q_α is an upper triangular matrix with all diagonal entries equal to one. Therefore, Q_α is invertible. This implies that T_α is invertible.

(ii) We now prove that $T_\alpha(S_\alpha^n) = S_0^n$ and $T_\alpha(RS_\alpha^n) = RS_0^n$. From the way in which T_α was defined, it follows that for every number x_0 such that $x_0^n + \Sigma_{j=1}^n \tilde{a}_j x_0^{n-j} = 0$, the number $y_0 = x_0 + \alpha$ satisfies $p(y_0) = y_0^n + \Sigma_{j=1}^n a_j y_0^{n-j} = 0$ and vice versa. Notice that $Re(x_0) < 0$ if and only if $Re(y_0) < \alpha$. Hence, $(a_1, a_2, \ldots, a_n) \in S_\alpha^n$ is equivalent to $T_\alpha(a_1, a_2, \ldots, a_n) = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \in S_0^n$. In other words, $T_\alpha(S_\alpha^n) = S_0^n$. The same proof holds for $T_\alpha(RS_\alpha^n) = RS_0^n$. □

Remark: For every $\sum_{j=1}^{n} \tilde{a}_j \vec{e}_j$, let $q(x) = x^n + \sum_{j=1}^{n} \tilde{a}_j x^{n-j}$. An elementary calculation shows that the inverse of T_α is defined by

$$T_\alpha^{-1}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)\tilde{Q}_\alpha + (\binom{n}{1}(-\alpha), \binom{n}{2}(-\alpha)^2, \ldots, \binom{n}{n}(-\alpha)^n)$$

where $\tilde{Q}_\alpha = (\tilde{q}_{ij})$ is an $n \times n$ matrix with $\tilde{q}_{ij} = \binom{n-i}{j-i}(-\alpha)^{j-i}$. Furthermore, T_α^{-1} satisfies the relations, $T_\alpha^{-1}(S_0^n) = S_\alpha^n$ and $T_\alpha^{-1}(RS_0^n) = RS_\alpha^n$.

Theorem 2.1.7. Given any $\alpha \in R$, the convex hulls of S_α^n and RS_α^n are both equal to the polygonal convex set formed by the intersection of the following half spaces:

$$\sum_{i=0}^{j} [\binom{n-i}{j-i}\alpha^{j-i}]a_i > 0 \qquad i = 1, 2, \ldots, n$$

Proof: Let T_α be the transformation defined in lemma 2.1.6. The theorem follows readily from the discussion preceding lemma 2.1.6 and (2.13). The same argument holds for RS_α^n. □

In terms of the usual standard basis of R^n, one may write $R_+^n = \{\sum_{k=1}^{n} r_k \vec{e}_k \mid r_k > 0\}$. Since $T_\alpha^{-1}(S_0^n) = S_\alpha^n$ and $co(S_0^n) = R_+^n$, it follows that

$$co(S_\alpha^n) = co(T_\alpha^{-1}(S_0^n)) = T_\alpha^{-1}(co(S_0^n))$$

$$= T_\alpha^{-1}(R_+^n) = \{T_\alpha^{-1}(\sum_{k=1}^{n} r_k \vec{e}_k) \mid r_k > 0\} = \{\sum_{j=1}^{n} \binom{n}{j}(-\alpha)^j \vec{e}_j +$$

$$\sum_{k=1}^{n} [\sum_{j=1}^{n} \tilde{q}_{kj} \vec{e}_j] r_k \mid r_k > 0\}$$

$$= \{\sum_{j=1}^{n} [\binom{n}{j}(-\alpha)^j + \sum_{k=1}^{n} \tilde{q}_{kj} r_k] \vec{e}_j \mid r_k > 0\} = \{\sum_{j=1}^{n} [\binom{n}{j}(-\alpha)^j +$$

$$\sum_{k=1}^{n} \binom{n-k}{j-k}(-\alpha)^{j-k} r_k] \vec{e}_j \mid r_k > 0\} \qquad (2.14)$$

where for each i, $\Sigma_{j=1}^{n} \tilde{q}_{kj} \vec{e}_j$ is the ith row of matrix $\tilde{Q}_\alpha$.
Hence, we obtain another way of expressing $co(S_\alpha^n)$.

Theorem 2.1.8. $co(S_\alpha^n)$ and $co(RS_\alpha^n)$ are both equal to the open
convex cone with vertex $\Sigma_{j=1}^{n} \binom{n}{j} (-\alpha)^j \vec{e}_j$, generated by the row
vectors of matrix $\tilde{Q}_\alpha = (\tilde{q}_{ij})$; explicityly, $\tilde{q}_{ij} = \binom{n-i}{j-i} (-\alpha)^{j-i}$.□
Note that the entries of the row vector $(\binom{n}{1} (-\alpha), \binom{n}{2} (-\alpha), \ldots,$
$\binom{n}{n} (-\alpha)^n)$ represent the coefficients of the polynomial $(x-\alpha)^n$
which has n identical roots $x = \alpha$.

Example: For $n = 2$ and a fixed real number α, $co(S_\alpha^2) = co(RS_\alpha^2) =$
$\{(a_1, a_2) \mid 2\alpha + a_1 > 0$ and $\alpha^2 + a_1\alpha + a_2 > 0$. It is the
shaded area in figure 2.1.

   As an easy generalization of theorem 2.1.8, one has

Theorem 2.1.9.   Given $\alpha \in R$, suppose Γ is a set of complex
numbers which satisfies

$$\{x \mid x < \alpha\} \subseteq \Gamma \subseteq \{z \mid Re(z) < \alpha\} \tag{2.15}$$

Let $\Pi(\Gamma)$ denote $\{\Sigma_{j=1}^{n} a_j \vec{e}_j \mid x^n + \Sigma_{j=1}^{n} a_j x^{n-j} = 0$ implies $x \in \Gamma\}$.
Then, $co(\Pi(\Gamma)) = co(S_\alpha^n) = co(RS_\alpha^n)$.

Proof: (2.15) implies that $co(RS_\alpha^n) \subseteq co(\Pi(\Gamma)) \subseteq co(S_\alpha^n)$. By
the relation $co(RS_\alpha^n) = co(S_\alpha^n)$, one concludes that $co(\Pi(\Gamma)) =$
$co(RS_\alpha^n) = co(S_\alpha^n)$.   □

   2.2 Criteria of S_α^n-Stabilizability of Hyperplanes
   First, let us prove the following fact.

Lemma 2.2.1. Given a hyperplane $H = \{(a_1, a_2, \ldots, a_n) \in R^n \mid A_0 + \Sigma_{j=1}^n A_j a_j = 0\}$ the following properties are equivalent.

(i) $H \cap R_+^n \neq \phi$

(ii) At least two of the numbers A_j, $j = 0, 1, 2, \ldots, n$ have strictly opposite signs.

Proof: (i) $\Rightarrow$ (ii). This follows easily from the observation that the sum $A_0 + \Sigma_{j=1}^n A_j a_j$ will never equal zero for any $(a_1, a_2, \ldots, a_n) \in R_+^n$ if all A_j, $j = 0, 1, 2, \ldots, n$ are of the same sign.

(ii) $\Rightarrow$ (i). Suppose that there exist $i \neq j$ such that $A_i A_j < 0$. Without loss of generality, let us assume $A_0 \geq 0$. Let $\Delta = \{j \mid A_j < 0\}$ and $\hat{\Delta} = \{j \mid A_j > 0\}$. Then, $H \cap R_+^n \neq \phi$, since $H \cap R_+^n$ contains at least the following point

$$\vec{z} = \Sigma_{j \in \Delta} \left[\frac{\Sigma_{j \in \hat{\Delta}} A_j}{\Sigma_{j \in \Delta} A_j} \right] \vec{e}_j + \Sigma_{j \in \hat{\Delta}} \vec{e}_j \qquad \square$$

Using this lemma, a criterion for determining S_α^n-stabilizability for hyperplanes can be easily formulated.

Theorem 2.2.2. Given a hyperplane $H = \{(a_1, a_2, \ldots, a_n) \in R^n \mid A_0 + \Sigma_{j=1}^n A_j a_j = 0\}$, let $h(x)$ be the polynomial $\Sigma_{j=1}^n A_{n-j} \binom{n}{n-j} x^{n-j}$. Let $\tilde{A}_i = \Sigma_{j=0}^n A_j \binom{n-i}{j-i} \alpha^{j-i}$ $i = 0, 1, 2, \ldots, n$. Then, the following statements are equivalent.

(i) H is RS_α^n-stabilizable.

(ii) H is S_α^n-stabilizable.

(iii) At least two of the numbers $\tilde{A}_i$, $i = 0, 1, 2, \ldots, n$, are of strictly opposite signs.

(iv)   There exist nonnegative integers $j \neq k$ such that $[h^{(j)}(-\alpha)][h^{(k)}(-\alpha)] < 0$ where $h^{(j)}(-\alpha)$ and $h^{(k)}(-\alpha)$ are respectively, the jth and kth derivative of $h(x)$ at $x = -\alpha$.

Proof: (i) $\Leftrightarrow$ (ii).   Recall that S_α^n and RS_α^n are open connected sets in R^n. Hence, by lemma 2.1.2, $H \cap S_\alpha^n \neq \phi$ if and only if $H \cap co(S_\alpha^n) \neq \phi$. Also, $H \cap RS_\alpha^n \neq \phi$ if and only if $H \cap co(RS_\alpha^n) \neq \phi$. Since $co(RS_\alpha^n) = co(S_\alpha^n)$, one concludes that the property $H \cap S_\alpha^n \neq \phi$ is equivalent to $H \cap RS_\alpha^n \neq \phi$.

(ii) $\Leftrightarrow$ (iii).   By lemma 2.1.2, we know that the property $H \cap S_\alpha^n \neq \phi$ is equivalent to

$$H \cap co(S_\alpha^n) \neq \phi \tag{2.16}$$

Recall from (2.14) that,

$$co(S_\alpha^n) = \{\Sigma_{j=1}^n [\,(\tbinom{n}{j})(-\alpha)^j + \Sigma_{i=1}^n (\tbinom{n-i}{j-i})(-\alpha)^{j-i} r_i]\vec{e}_j \,|\, r_i > 0\}$$

Thus, (2.16) is true if and only if there exist $r_i > 0$, $i = 1, 2, \ldots, n$ such that

$$A_0 + \sum_{j=1}^n [(\tbinom{n}{j})(-\alpha)^j + \sum_{i=1}^n (\tbinom{n-i}{j-i})(-\alpha)^{j-i} r_i]A_j = [A_0 + \sum_{j=1}^n (\tbinom{n}{j})(-\alpha)^j]$$

$$+ \sum_{i=1}^n [\sum_{j=1}^n (\tbinom{n-i}{j-i})(-\alpha)^{j-i} A_j] r_i = \tilde{A}_0 + \sum_{i=1}^n \tilde{A}_i r_i = 0 \tag{2.17}$$

By lemma 2.2.1, the existence of such r_i's which satisfy (2.17) is equivalent to the property that at least two of the numbers, $\tilde{A}_j$, $j = 0, 1, 2, \ldots, n$ are of strictly opposite signs.

(iii) $\Leftrightarrow$ (iv). This equivalence follows readily from the observation

$$h^{(i)}(-\alpha) = \frac{n!}{i!} \tilde{A}_i \quad \text{for each } i = 0,1,2,\ldots,n \qquad \Box$$

A special case worth mentioning is when $\alpha = 0$.

Corollary 2.2.3. A hyperplane $H = \{(a_1, a_2, \ldots, a_n) \in R^n \mid A_0 + \Sigma_{i=1}^{n} A_i a_i = 0\}$ is S_0^n-stabilizable if and only if $A_j A_k < 0$ for some integers $j \neq k$ where $0 \leq j, k \leq n$.

Proof: The proof follows readily from the fact that $\tilde{A}_i = A_i$ for all i, when $\alpha = 0$. $\qquad \Box$

It is interesting to note that the equivalence between (i) and (ii) of theorem 2.2.2 means that on any S_α^n-stabilizable hyperplane not only can one find points $(a_1, a_2, \ldots, a_n) \in H$ such that $x^n + \Sigma_{j=1}^{n} a_j x^{n-j} = 0$ has only roots with real parts smaller than α but one also can find points $(a_1, a_2, \ldots, a_n) \in H$ for which all the roots of the polynomial equation $x^n + \Sigma_{j=1}^{n} a_j x^{n-j} = 0$ are real, distinct and less than α. In fact, theorem 2.2.2 can be generalized to the following:

Theorem 2.2.4. Let Γ be a set of complex numbers satisfying the condition

$$\{x \mid x < \alpha\} \subseteq \Gamma \subseteq \{z \mid \text{Re}(z) < \alpha\}$$

Let $\Pi(\Gamma) = \{(a_1, a_2, \ldots, a_n) \in R^n \mid x^n + \Sigma_{j=1}^{n} a_j x^{n-j} = 0 \text{ implies } x \in \Gamma\}$. Then, the property $H \cap \Pi(\Gamma) \neq \phi$ is equivalent to $H \cap RS_\alpha^n \neq \phi$.

Proof: Observe that $RS_\alpha^n \subset H(\Gamma)$. Therefore, $H \cap RS_\alpha^n \neq \phi$
implies $H \cap H(\Gamma) \neq \phi$. Conversely, if $H \cap H(\Gamma) \neq \phi$, then
$H \cap co(H(\Gamma)) \neq \phi$. Since, by theorem 2.1.10, $co(H(\Gamma)) = co(RS_\alpha^n)$,
it follows that $H \cap co(RS_\alpha^n) \neq \phi$. This implies that
$H \cap RS_\alpha^n \neq \phi$, because RS_α^n is open connected in R^n.

2.3 Bounds of Stabilizability

Let $H = \{(a_1, a_2, \ldots, a_n) \in R^n | A_0 + \Sigma_{j=1}^n A_j a_j = 0\}$ be a
hyperplane in R^n. Since $S_\alpha^n \subset S_\beta^n$ if and only if $\alpha < \beta$, it is
obvious that there exists always an α_0 such that $H \cap S_\alpha^n \neq \phi$
for all $\alpha > \alpha_0$. Hence, the set $\{\alpha | H \cap S_\alpha^n \neq \phi\}$ is always
unbounded from above. The interesting question is "what is
the $\inf\{\alpha | H \cap S_\alpha^n \neq \phi\}$?" The next theorem gives the answer.

Theorem 2.3.1. Given a hyperplane $H = \{(a_1, a_2, \ldots, a_n) \in R^n |$
$A_0 + \Sigma_{j=1}^n A_j a_j = 0\}$ in R^n, let $h(x) = \Sigma_{j=0}^n A_{n-j} \binom{n}{n-j} x^{n-j}$ and
$k = \max\{j | A_j \neq 0\}$. Also, let $\eta = -\max\{\alpha | \Pi_{i=0}^k h^{(i)}(\alpha) = 0\}$.
Then, $\inf\{\alpha | H \cap S_\alpha^n \neq \phi\} = \eta$.

Proof: Observe that $h(x)$ and its derivatives $h^{(j)}(x)$, $j = 1$,
$2, \ldots, n$, have the same leading coefficient A_k. Without loss of
generality, we assume that $A_k > 0$. Then, for every $\alpha < \eta$,
$h^{(i)}(-\alpha) > 0$ for all $i = 0, 1, 2, \ldots, n$. In view of theorem 2.2.2,
this implies that $H \cap S_\alpha^n = \phi$ for every $\alpha < \eta$. Hence,

$$\inf\{\alpha | H \cap S_\alpha^n \neq \phi\} \geq \eta \qquad (2.18)$$

Conversely, by the definition of η, there exists some positive
integer $\hat{i} \leq k-1$ such that $h^{(\hat{i})}(-\eta) = 0$. Also, since $h^{(\hat{i})}(x)$

is a polynomial of positive degree with positive leading coefficient, there exists an $\epsilon > 0$ small enough such that for every $\alpha \in (\eta, \eta+\epsilon)$ either $h^{(\hat{i}+1)}(-\alpha) > 0$ or $h^{(\hat{i}+1)}(-\alpha) < 0$. In the latter case, since $h^{(k)}(-\alpha) = (n!/k!)A_k > 0$ (we assumed $A_k > 0$), one has $[h^{(\hat{i}+1)}(-\alpha)][h^{(k)}(-\alpha)] < 0$ for all $\alpha \in (\eta, \eta+\epsilon)$. In the first case, one has $h^{(\hat{i}+1)}(-\alpha) > 0$ for all $\alpha \in (\eta, \eta+\epsilon)$ and $h^{(\hat{i})}(-\alpha) = 0$. Consequently, $h^{(\hat{i})}(-\alpha) < 0$ for all $\alpha \in (\eta, \eta+\epsilon)$. Therefore, $h^{(k)}(-\alpha) \, h^{(\hat{i})}(-\alpha) < 0$ for all $\alpha \in (\eta, \eta+\epsilon)$. By theorem 2.2.2, both cases imply that $H \cap S_\alpha^n \neq \phi$ for all $\alpha \in (\eta, \eta+\epsilon)$.

Using (2.18), one concludes that $\inf\{\alpha \mid H \cap S_\alpha^n \neq \phi\} = \eta$. $\square$
With η defined as above, the assertion of theorem 2.3.1 can be restated as follows: "A hyperplane H is S_α^n-stabilizable if and only if $\alpha > \eta$."

Remark: Recall that in the introduction, we noticed that if $m = n$, then for any $\alpha \in R$ there exist always parameters k_i, $i = 1, 2, \ldots, n$, such that all the roots of the equation $p(x) + \Sigma_{j=1}^n k_i p_i(x) = 0$ (*) satisfy the inequality $\text{Re}(x) < \alpha$. In other words, one may "push" all the roots of (*) toward the left side of the complex plane as far as one wishes by the proper choice of k_i's. The theorem above shows that this is not true for the case $m = n-1$. In fact, the theorem describes the "limit" which can be achieved for all the possible choices of k_i's.

Before we prove the next theorem, let us divert for a moment to prove the following property of polynomials.

Lemma 2.3.2. Suppose that λ is the only one (counting the multiplicity) positive real root of a polynomial f whose derivative f' has at most one positive real root. Then, $f'(x) \neq 0$ for all $x > \lambda$.

Proof: Assume the contrary: there exists $x_0 > \lambda$ such that $f'(x_0) = 0$. Then, either $f''(x_0) = 0$ or $f''(x_0) > 0$ or $f''(x_0) < 0$. We will show in the following that all these cases lead to contradictions. Without loss of generality, we assume that f has a positive leading coefficient.

(i) If $f''(x_0) = 0$, then, by the assumption $f'(x_0) = 0$, one finds that $x = x_0$ is a positive double root of $f'(x) = 0$. This contradicts the hypothesis that f' has at most one positive real root.

Note that λ is the only one positive real root of $f(x)$ and that $f(x)$ has positive leading coefficient. Hence, $f(x) > 0$ for all $x > \lambda$. Therefore, there exists $\varepsilon > 0$ small enough such that $\lambda + \varepsilon < x_0$ and for all $x \in (\lambda, \lambda + \varepsilon)$

$$f'(x) > 0$$

(ii) Suppose that $f''(x_0) > 0$. Then, from the assumption $f'(x_0) = 0$, it follows that there exists a $\delta > 0$ small enough such that $\lambda + \varepsilon < x_0 - \delta$ and $f'(x) < 0$ for all $x \in (x_0 - \delta, x_0)$. Since $f'(\lambda + \frac{\varepsilon}{2}) > 0$, and $f'(x_0 - \delta/2) < 0$, by the intermediate value theorem, there exists x_1 where $\lambda < \lambda + \varepsilon/2 < x_1 < x_0 - \delta/2$ such that $f'(x_1) = 0$. Therefore, f' contains two positive real roots $x = x_1$ and $x = x_0$, a contradiction.

(iii) Suppose that $f''(x_0) < 0$. Then, since $f'(x_0) = 0$, there exists a $\delta > 0$ such that $f'(x) < 0$ for all $x \in (x_0, x_0 + \delta)$. Observe that f' has positive leading coefficient. Hence, there exists $\beta > x_0 + \delta$ such that $f'(x) > 0$ for all $x \geq \beta$. Consequently, $f'(x_0 + \delta/2) \cdot f'(\beta) < 0$. By the intermediate value theorem, there exists x_2 where $x_0 < x_0 + \delta/2 < x_2 < \beta$ such that $f'(x_2) = 0$, a contradiction. This completes the proof. □

Theorem 2.3.3. Given a hyperplane $H = \{(a_1, a_2, \ldots, a_n) \in R^n \mid A_0 + \Sigma_{j=1}^n A_j a_j = 0\}$, let $k = \max\{j \mid A_j \neq 0\}$. Let $h(x) = \Sigma_{j=0}^n A_j \binom{n}{j} x^j$. Let $\eta = \inf\{\alpha \mid H \cap S_\alpha^n \neq \phi\}$. If there exists a nonnegative integer $\ell \leq k-1$ such that $A_j < 0$ for all $j \leq \ell$ and $A_j \geq 0$ for all $j > \ell$, then $\eta = -\max\{\alpha \mid h(\alpha) = 0\}$.

Proof: Obviously, $-\max\{\alpha \mid h(\alpha) = 0\} \geq -\max\{\alpha \mid \Pi_{j=0}^k h^{(j)}(\alpha) = 0\} = \eta$. Conversely, recall that $h(x) = \Sigma_{j=0}^n A_j \binom{n}{j} x^j$. The hypothesis implies that there is exactly one variation of signs among the coefficients of $h^{(j)}(x)$ for each $0 \leq j \leq \ell-1$ and no variation of signs of coefficients of $h^{(j)}(x)$ for $j \geq \ell$. Therefore, by Descartes Rule, the polynomial $h^{(j)}$ has exactly one positive real root if $j \leq \ell-1$ and no positive real root if $j \geq \ell$. It follows that

$$-\max\{\alpha \mid \Pi_{j=0}^{\ell-1} h^{(j)}(\alpha) = 0\} = -\max\{\alpha \mid \Pi_{j=0}^{k} h^{(j)}(\alpha) = 0\} = \eta$$

Furthermore, by applying lemma 2.3.2, to h, h'; $h', h'', \ldots,$ $h^{(\ell-2)}, h^{(\ell-1)}$, one obtains that, for each $0 \leq j \leq \ell-1$,

$$\max\{\alpha \mid h^{(j)}(\alpha) = 0\} \leq \max\{\alpha \mid h^{(j-1)}(\alpha) = 0\}$$

Consequently,

$$\max\{\alpha \mid \prod_{j=0}^{\ell-1} h^{(j)}(\alpha) = 0\} = \max\{\alpha \mid h(\alpha) = 0\}$$

Hence,

$$\eta = -\max\{\alpha \mid h(\alpha) = 0\} \qquad \square$$

Remark: Given a hyperplane $H \in R^n$, suppose $-\inf\{\alpha \mid H \cap S_\alpha^n \neq \phi\} = \eta$. From the point of view of applications it is important to actually find the points in $H \cap S_\alpha^n \neq \phi$ if $\alpha > \eta$. Unfortunately, we do not have a general method for such a task. Following is a special case for which we do have a method. Suppose H is a hyperplane in R^n such that $\eta = -\max\{\alpha \mid h(\alpha) = 0\}$. Then, $h(-\eta) = 0$. Recall that $h(x) = \Sigma_{j=0}^n A_j \binom{n}{j} x^j$. Therefore, $\Sigma_{j=0}^n A_j \binom{n}{j}(-\eta)^j = 0$. This implies that

$$(\binom{n}{1}(-\eta), \binom{n}{2}(-\eta)^2, \ldots, \binom{n}{n}(-\eta)^n) \in H$$

Note also that

$$(\binom{n}{1}(-\eta), \binom{n}{2}(-\eta)^2, \ldots, \binom{n}{n}(-\eta)^n) \in \overline{RS_\eta^n}$$

since the polynomial $x^n + \Sigma_{j=1}^n \binom{n}{j}(-\eta)^j x^{n-j} = 0$ has only identical roots $x = \eta$. Hence,

$$(\binom{n}{1}(-\eta), \binom{n}{2}(-\eta)^2, \ldots, \binom{n}{n}(-\eta)^n) \in H \cap \overline{RS_\eta^n}$$

We remark that this method can be applied to the kind of hyperplane described in theorem 2.3.3.

In the following, we illustrate the results which we obtained in this chapter by an example chosen from Anderson et al. [1].

Example: Consider a system

$$(E) \quad x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 5 & -1 \\ -1 & -1 & 0 \end{pmatrix} x, \quad u = Ky = [v,w]y$$

Then, the characteristic polynomial of the closed loop system is

$$s^3 + vs^2 + (w-5v-13)s + w = 0 \tag{2.19}$$

Let H denote the set of all the coefficients vectors arise from all possible choice of $K = [v,w]$, i.e.,

$$H = \{(a_1,a_2,a_3) \mid a_1 = v, \ a_2 = w-5v-13, \ a_3 = w, \text{ for some}$$
$$v,w \in R\} = \{(a_1,a_2,a_3) \mid 5a_1 + a_2 - a_3 + 13 = 0\}.$$

Note that H forms a hyperplane in R^3. Since $A_1 = 5 > 0$ and $A_3 = -1 < 0$, by corollary 2.2.3, the system (E) is S_0-stabilizable. In other words, there exists a $K \in R^{1 \times 2}$ such that all the roots of (2.19) have negative real parts.

Consider the polynomial $h(x)$ defined in theorem 2.3.3. For system (E), $h(x) = -x^3 + 3x^2 + 15x + 13$. By theorem 2.3.3, one concludes that

$$\inf\{\alpha \mid H \cap S_\alpha^3\} = -\max_\alpha\{\alpha \mid h(\alpha) = 0\} \approx -5.91$$

This means that no matter how we choose $K = [v,w]$, there exists at least one root of (2.19) with real part greater than or equal to the number $-\max\{\alpha \in R \mid h(\alpha) = 0\} \approx -5.91$.

CHAPTER III

$C_{\alpha,\beta}^n$-STABILIZABLE HYPERPLANES

Given $\alpha < \beta$, let $C_{\alpha,\beta}^n$ denote the set of all the points $(a_1, a_2, \ldots, a_n) \in R^n$ for which the polynomial equation $x^n + \sum_{j=1}^n a_j x^{n-j} = 0$ has only roots inside the circle centered at $((\alpha+\beta)/2, 0)$ having radius $(\beta-\alpha)/2$. Let $RC_{\alpha,\beta}^n$ be the set

$\{(a_1, a_2, \ldots, a_n) \in R^n \mid$ all the roots of $x^n + \sum_{j=1}^n a_j x^{n-j} = 0$

are real, distinct and strictly betwen α and $\beta\}$.

A hyperplane $H \subset R^n$ is said to be stabilizable with respect to the circle $\{z \mid |z - (\beta+\alpha)/2| = (\beta-\alpha)/2\}$ if $H \cap C_{\alpha,\beta}^n \neq \phi$. H is stabilizable with respect to the interval $\{x \mid \alpha < x < \beta\}$ if $H \cap RC_{\alpha,\beta}^n \neq \phi$. In this chapter we investigate the following two problems:

(i)   What are the $co(RC_{\alpha,\beta}^n)$ and $co(C_{\alpha,\beta}^n)$?

(ii)   How can one characterize the hyperplanes which are stabilizable with respect to a given circle and the hyperplanes which are stabilizable with respect to a given interval?

Particularly interesting from the stability point of view, is the case of the unit circle centered at the origin. In fact, the question of the stabilizability of a hyperplane with respect to the unit circle centered at the origin arises from the problem of direct output feedback stabilization of a sampled data control system. The well known Schur-Cohen

criterion which characterizes $C_{-1,1}^n$ consists of a system of nonlinear inequalities. Hence, it may be a difficult task to answer questions (i) and (ii) directly in terms of those inequalities. Instead, by an invertible linear transformation constructed in section 1, we shall reduce (i) and (ii) to problems solved in chapter 2. As a main result, we prove that $co(C_{\alpha,\beta}^n)$ and $co(RC_{\alpha,\beta}^n)$ are both equal to the interior of the simplex generated by the points $(q_{i1}, q_{i2}, \ldots, q_{in}) \in R^n$ $i = 0,1,2,\ldots,n$ where q_{ij} are the coefficients of the polynomial $(x-\alpha)^{n-i}(x-\beta)^i = x^n + \Sigma_{j=1}^n q_{ij}x^{n-j}$.

3.1 The Convex Hulls, $co(C_{\alpha,\beta}^n)$ and $co(RC_{\alpha,\beta}^n)$

We will denote the standard basis of R^m by $\{\vec{e}_j^m\}_{j=1}^m$. For brevity, if there is no confusion of the dimension, we will simply write $\{\vec{e}_j\}_{j=1}^m$. First, let us extend lemma 2.1.5 to the following sets

$$\widetilde{S} = \{ \sum_{j=0}^n a_j \vec{e}_{j+1}^{n+1} \mid a_0 > 0, \quad \sum_{j=0}^n a_j x^{n-j} = 0 \text{ implies } Re(x) < 0\}$$

$$R\widetilde{S} = \{ \sum_{j=0}^n a_j \vec{e}_{j+1}^{n+1} \mid a_0 > 0, \text{ all the roots } x \text{ of } \sum_{j=0}^n a_j x^{n-j} = 0$$

$$\text{are real, distinct and } x < 0\}$$

Lemma 3.1.1. $co(\widetilde{S}) = co(R\widetilde{S}) = R_+^{n+1}$.

Proof: Obviously, by Hurwitz criterion and the convexity property of R_+^{n+1} one has $co(R\widetilde{S}) \subseteq co(\widetilde{S}) \subseteq R_+^{n+1}$.

Conversely, if $\Sigma_{j=0}^n a_j \vec{e}_{j+1}^{n+1} \in R_+^{n+1}$, then $\Sigma_{j=1}^n (a_j/a_0)\vec{e}_j^n \in R_+^n$. Recall that $R_+^n = co(RS_0^n)$. Therefore, $\Sigma_{j=1}^n (a_j/a_0)\vec{e}_j^n$ can be

expressed as a convex combination of points from RS_0^n. In other words, there exist points $\sum_{j=1}^n (b_{ij}/a_0) \dot{e}_j^n \in RS_0^n$ and parameters $k_i > 0$ satisfying $\sum_i k_i = 1$, such that

$$\sum_{j=1}^n (a_j/a_0) \tilde{e}_j^n = \sum_{j=1}^n [\sum_{i=1}^n k_i (b_{ij}/a_0)] \tilde{e}_j^n \text{ or equivalently,}$$

$$\sum_{j=0}^n a_j \dot{e}_{j+1}^{n+1} = a_0 \dot{e}_1^{n+1} + \sum_{j=1}^n [\sum_{i=1}^n k_i b_{ij}] \dot{e}_{j+1}^{n+1} \tag{3.1}$$

Observe that for each i, the equations $x^n + \sum_{j=1}^n (b_{ij}/a_0) x^{n-j} = 0$ and $a_0 x^n + \sum_{j=1}^n b_{ij} x^{n-j} = 0$ have the same roots. Hence, from $\sum_{j=1}^n (b_{ij}/a_0) \dot{e}_j^n \in RS_0^n$, one obtains $a_0 \dot{e}_1^{n+1} + \sum_{j=0}^n b_{ij} \dot{e}_{j+1}^{n+1} \in R\tilde{S}$. Therefore, by (3.1), one concludes that $\sum_{j=1}^n a_j \dot{e}_{j+1}^{n+1} \in co(R\tilde{S})$ for any $\sum_{j=0}^n a_j \dot{e}_{j+1}^{n+1} \in R_+^{n+1}$. This implies $R_+^{n+1} \subseteq co(R\tilde{S})$. []

For any fixed $\alpha < \beta$, define

$$\tilde{C}_{\alpha,\beta}^n = \{\sum_{j=0}^n a_j \dot{e}_{j+1}^{n+1} \in R^{n+1} | \; a_0 > 0 \quad \sum_{j=1}^n a_j x^{n-j} = 0 \text{ implies}$$

$$|x - (\alpha+\beta)/2| < (\beta-\alpha)/2\}$$

and

$$R\tilde{C}_{\alpha,\beta}^n = \{\sum_{j=0}^n a_j \dot{e}_{j+1}^{n+1} \in R^{n+1} | a_0 > 0, \text{ all the roots of}$$

$$\sum_{j=0}^n a_j x^{n-j} = 0 \text{ are real, distinct, and striclty between}$$

α and $\beta\}$.

Let $T_{\alpha,\beta}$ be the transformation on R^{n+1} such that for each $\sum_{j=0}^n a_j \dot{e}_{j+1}^{n+1} \in R^{n+1}$

$$T_{\alpha,\beta}(\sum_{j=0}^n a_j \dot{e}_{j+1}^{n+1}) = \sum_{j=0}^n \tilde{a}_j \dot{e}_{j+1}^{n+1}$$

where $\tilde{a}_j$, $j = 0,1,2,\ldots,n$ are obtained from the following polynomial expansion:

$$\sum_{j=0}^{n} a_j (s-\alpha)^{n-j} (s-\beta)^j = \sum_{j=0}^{n} \tilde{a}_j s^{n-j}$$

For $\alpha = 1$ and $\beta = 1$, it is well known that $T_{-1,1}$ is a linear transformation which maps $\tilde{S}$ onto $C_{-1,1}^n$ (Hermite [1], Marden [1]). The matrix representation $Q_{-1,1}$ of $T_{-1,1}$, due to its application to the stability of a discrete system, has been investigated by many researchers such as Power [1], Fielder [1], Halijak and Moe [1], Jury and Chan [1], Duffin [1], etc. Barnett [1] has studied the linear transformation T induced by a more general bilinear transformation $z = \frac{\gamma s - \alpha}{\delta s - \beta}$ and showed that the entries of the matrix representation of T can be obtained by an interesting algorithm. For our purpose, we need the following special case of Barnett's (i.e., $\delta = 1$ and $\gamma = 1$).

<u>Lemma 3.1.2.</u> For any $\alpha < \beta$, $T_{\alpha,\beta}$ is an invertible linear transformation on R^{n+1} which satisfies $T_{\alpha,\beta}(\tilde{S}) = \tilde{C}_{\alpha,\beta}^n$ and $T_{\alpha,\beta}(R\tilde{S}) = R\tilde{C}_{\alpha,\beta}^n$.

<u>Proof:</u> We divide the proof into three parts:

(i) $T_{\alpha,\beta}$ is linear.

For any $\sum_{j=0}^{n} a_j \vec{e}_{j+1}^{n+1}$ and $\sum_{j=0}^{n} b_j \vec{e}_{j+1}^{n+1}$, let $\sum_{j=0}^{n} \tilde{c}_j \vec{e}_{j+1}^{n+1} = T_{\alpha,\beta}(\sum_{j=0}^{n} (a_j + b_j) \vec{e}_{j+1}^{n+1})$. By the definition of $T_{\alpha,\beta}$, the $\tilde{c}_j$, $j = 0,1,\ldots,n$, are obtained from the following polynomial expansion

$$\sum_{j=0}^{n} \tilde{c}_j s^{n-j} = \sum_{j=0}^{n} (a_j + b_j)(s-\alpha)^{n-j}(s-\beta)^j \qquad (3.2)$$

If $\sum_{j=0}^{n} \tilde{a}_j \tilde{e}_{j+1}^{n+1}$ and $\sum_{j=0}^{n} \tilde{b}_j \tilde{e}_{j+1}^{n+1}$ are the images of $\sum_{j=0}^{n} a_j \tilde{e}_{j+1}^{n+1}$

and $\sum_{j=0}^{n} b_j \tilde{e}_{j+1}^{n+1}$ under $T_{\alpha,\beta}$, respectively, then the right hand

side of (3.2) equals $\sum_{j=0}^{n} (\tilde{a}_j + \tilde{b}_j) s^{n-j}$. Thus, $\tilde{c}_j = \tilde{a}_j + \tilde{b}_j$,

$j = 0,1,2,\ldots,n$. Hence,

$$T_{\alpha,\beta}(\sum_{j=0}^{n} (a_j+b_j) \tilde{e}_{j+1}^{n+1}) = T_{\alpha,\beta}(\sum_{j=0}^{n} a_j \tilde{e}_{j+1}^{n+1}) + T_{\alpha,\beta}(\sum_{j=0}^{n} b_j \tilde{e}_{n+1}^{n+1})$$

(ii) $T_{\alpha,\beta}$ is invertible.

To show that $T_{\alpha,\beta}$ is invertible, since $T_{\alpha,\beta}$ is linear,

it suffices to show that $T_{\alpha,\beta}$ is onto. In other words, for

any $\sum_{j=0}^{n} \tilde{a}_j \tilde{e}_{j+1}^{n+1}$ we need to find $\sum_{j=0}^{n} a_j \tilde{e}_{j+1}^{n+1}$ such that

$$\sum_{j=0}^{n} a_j (s-\alpha)^{n-j} (s-\beta)^j = \sum_{j=0}^{n} \tilde{a}_j s^{n-j} \qquad (3.3)$$

Substitute $(\beta z-\alpha)/(z-1)$ for s in (3.3) and multiply both sides

by $(z-1)^n$. Then, (3.3) becomes

$$\sum_{j=0}^{n} a_j z^{n-j} = \sum_{j=0}^{n} \tilde{a}_j (\beta z-\alpha)^{n-j} (z-1)^j \qquad (3.4)$$

In view of this, one may simply choose a_j's to be the coeffi-

cients of the polynomial expansion of the right hand side of

(3.4). This proves that $T_{\alpha,\beta}$ is onto. Hence, $T_{\alpha,\beta}$ is

invertible.

(iii) $T_{\alpha,\beta}(\tilde{S}) = \tilde{C}_{\alpha,\beta}^n$ and $T_{\alpha,\beta}(R\tilde{S}) = R\tilde{C}_{\alpha,\beta}^n$.

Consider the 1-1 transformation

$$s \longrightarrow z = \frac{s-\alpha}{s-\beta} \qquad (3.5)$$

which maps the interior of the circle $\{s \mid |s-(\alpha+\beta)/2| = (\beta-\alpha)/2\}$ onto the open left half plane $\{z \mid \mathrm{Re}(z) < 0\}$. In particular, (3.5) maps the interval $\{s \mid \alpha < s < \beta\}$ onto the negative real axis. Hence, for any nth degree polynomial $p(z)$, the statement: "all the roots of $p(z) = 0$ are in $\{z \mid \mathrm{Re}(z) < 0\}$" is equivalent to the statement: "all the roots of $\bar{p}(s) = (s-\beta)^n p(\frac{s-\alpha}{s-\beta}) = 0$ are in $\{s \mid |s-(\alpha+\beta)/2| < (\beta-\alpha)/2\}$." Furthermore, $\{z \mid p(z) = 0\} \subset \{z \mid \mathrm{Re}(z) < 0\}$ if and only if $\{s \mid \bar{p}(s) = 0\} \subset \{s \mid \alpha < s < \beta\}$. Observe that for any $p(z) = \sum_{j=0}^{n} a_j x^{n-j}$, $(s-\beta)^n p(\frac{s-\alpha}{s-\beta}) = \sum_{j=0}^{n} a_j (s-\alpha)^{n-j}(s-\beta)^j = \sum_{j=0}^{n} \tilde{a}_j s^{n-j}$. Recall that this was how the image of $\sum_{j=0}^{n} a_j \vec{e}_{j+1}^{n+1}$ under $T_{\alpha,\beta}$ was defined. Hence, $T_{\alpha,\beta}$, determines a one to one correspondence between the Hurwitz polynomials and polynomials whose roots lie inside the circle $\{s \mid |s-(\alpha+\beta)/2| = (\beta-\alpha)/2\}$. Similarly, $T_{\alpha,\beta}$ determines a 1-1 correspondence between the polynomials which have only distinct negative real roots and the polynomials of which all the roots are distinct real and strictly between α and β. If the leading coefficient a_0 of a Hurwitz polynomial is negative, then $a_j < 0$, $j = 1,2,\ldots,n$. Hence, $\tilde{a}_0 = \sum_{j=0}^{n} a_j < 0$ which means that $T_{\alpha,\beta}(\sum_{j=0}^{n} a_j \vec{e}_{j+1}^{n+1}) = \sum_{j=0}^{n} \tilde{a}_j \vec{e}_{j+1}^{n+1} \in \tilde{C}_{\alpha,\beta}^n$. Therefore, $T_{\alpha,\beta}(\tilde{S}) = \mathrm{co}(\tilde{C}_{\alpha,\beta}^n)$ and $T_{\alpha,\beta}(R\tilde{S}) = R\tilde{C}_{\alpha,\beta}^n$. $\square$

Following a similar technique as Barnett [1], we compute explicitly the matrix representation $Q_{\alpha,\beta}$ of $T_{\alpha,\beta}$ relative to the usual standard basis of R^{n+1}.

Note that $Q_{\alpha,\beta}$ is an n+1 by n+1 matrix such that

$$T_{\alpha,\beta}(\sum_{j=0}^{n} a_j \vec{e}_{j+1}^{n+1}) = (a_0, a_1, \ldots, a_n) Q_{\alpha,\beta}.$$

Hence, from the definition of $T_{\alpha,\beta}$, for each fixed i, the row $(q_{i0}, q_{i1}, \ldots, q_{in})$ of $Q_{\alpha,\beta}$ represents the coefficients of the polynomial

$$(s-\alpha)^{n-i}(s-\beta)^{i} \tag{3.6}$$

By a simple combinatorial argument, one may expand (3.6) as

$$(s-\alpha)^{n-i}(s-\beta)^{i} = \sum_{j=0}^{n} [(-1)^{j} \sum_{k=0}^{j} \binom{i}{k} \binom{n-i}{j-k} \alpha^{j-k} \beta^{k}] s^{n-j} \tag{3.7}$$

Hence, for each $0 \le i \le n$ and $0 \le j \le n$, one obtains

$$q_{ij} = (-1)^{j} \sum_{k=0}^{j} \binom{i}{k} \binom{n-i}{j-k} \alpha^{j-k} \beta^{k} \tag{3.8}$$

In particular, from (3.7), it is easy to see that $q_{i0} = 1$, $i = 0,1,2,\ldots,n$.

We remark that the determinant of $Q_{\alpha,\beta}$ is equal to $(\alpha-\beta)^{\frac{n(n+1)}{2}}$ (see appendix II).

Now consider a transformation $\tilde{T}_{\alpha,\beta}$ on R^{n+1} defined by

$$\tilde{T}_{\alpha,\beta}(\sum_{j=0}^{n} \tilde{a}_{j} \vec{e}_{j+1}^{n+1}) = \sum_{j=0}^{n} a_{j} \vec{e}_{j+1}^{n+1}$$

where the a_j, $j = 0,1,2,\ldots,n$ are obtained from the following polynomial expansion:

$$\sum_{j=0}^{n} \tilde{a}_{j} (\beta z - \alpha)^{n-j} (z-1)^{j} = \sum_{j=0}^{n} a_{j} z^{n-j}$$

By the transformation $z \longrightarrow s = \frac{\beta z - \alpha}{z-1}$ and with the same technique used in establishing lemma 3.1.2, one can show that $\tilde{T}_{\alpha,\beta}$ is a linear invertible transformation on R^{n+1} which

satisfies the relations $\tilde{T}_{\alpha,\beta}(\tilde{C}^n_{\alpha,\beta}) = R\tilde{S}$ and $\tilde{T}_{\alpha,\beta}(\tilde{C}^n_{\alpha,\beta}) = \tilde{S}$.
Furthermore, one finds that the entries $\tilde{q}_{ij}$ of the matrix
representation $\tilde{Q}_{\alpha,\beta}$ of $\tilde{T}_{\alpha,\beta}$ are

$$\tilde{q}_{ij} = (-1)^j \sum_{k=0}^{j} \binom{i}{k}\binom{n-i}{j-k}\alpha^{j-k}\beta^{n-i-j+k}$$

We remark that one may use the algorithm proposed by Barnett [1]
to generate the entries of matrix $\tilde{Q}_{\alpha,\beta}$.

For the unit circle case, i.e., $\alpha = -1$ and $\beta = 1$, it is
interesting to note that the matrices $Q_{-1,1}$ and $\tilde{Q}_{-1,1}$ which
are associated with $T_{-1,1}$ and $\tilde{T}_{-1,1}$, respectively, are equal.
Duffin [1] and Jury and Chan [1] have computed the matrix
explicitly and have proved the following relations between the
entries of $Q_{-1,1}$

$$q_{ij} = q_{i-1,j} - q_{i-1,j-1} - q_{i,j-1} \qquad i \geq 1, \; j \geq 1 \qquad (3.10)$$

$$Q_{-1,1}Q_{-1,1} = (-2)^{n(n+1)} I$$

Obviously, (3.10) can be easily programmed on a computer to
generate the matrix $Q_{-1,1}$.

Now, recall that if T is an invertible affine transfor-
mation on R^m, then for any set $S \subset R^m$, $co(T(S)) = co(T(S))$.
Using this, we obtain the following proposition.

Proposition 3.1.3. For any fixed $\alpha < \beta$ and every n, the sets
$co(\tilde{C}^n_{\alpha,\beta})$ and $co(R\tilde{C}^n_{\alpha,\beta})$ are both equal to the polygonal convex
set in R^{n+1} formed by the intersection of the following half
spaces:

$$\sum_{i=0}^{n} [(-1)^j \sum_{k=0}^{j} \binom{n-i}{j-k}\binom{i}{k}\alpha^{j-k}\beta^{n-i-j+k}]a_i > 0 \qquad j = 0,1,2,\ldots,n.$$

__Proof:__ Recall that $\tilde{T}_{\alpha,\beta}(\tilde{C}^n_{\alpha,\beta}) = \tilde{S}$ and $\tilde{T}_{\alpha,\beta}(R\tilde{C}^n_{\alpha,\beta}) = R\tilde{S}$. From lemma 3.1.1, it follows that

$$\tilde{T}_{\alpha,\beta}(co(\tilde{C}^n_{\alpha,\beta})) = co(\tilde{T}_{\alpha,\beta}(\tilde{C}^n_{\alpha,\beta})) = co(\tilde{S}) = R^{n+1}_+$$

also,

$$\tilde{T}_{\alpha,\beta}(co(R\tilde{C}^n_{\alpha,\beta})) = co(\tilde{T}_{\alpha,\beta}(R\tilde{C}^n_{\alpha,\beta})) = co(R\tilde{S}) = R^{n+1}_+$$

Hence, $\Sigma^n_{j=0} a_j \vec{e}^{n+1}_{j+1} \in co(\tilde{C}^n_{\alpha,\beta}) = co(R\tilde{C}^n_{\alpha,\beta})$ if and only if $T_{\alpha,\beta}(\Sigma^n_{j=0} a_j \vec{e}^{n+1}_{j+1}) = (a_0, a_1, \ldots, a_n) Q_{\alpha,\beta} > 0$ or equivalently, $\Sigma^n_{i=0}[(-1)^j \Sigma^j_{k=0} \binom{n-i}{j-k}\binom{i}{k} \alpha^{j-k}\beta^{n-i-j+k}]a_i > 0$ for each $i = 0,1,2,\ldots,n$. □

Observe that, in terms of the usual basis vector of R^{n+1} the positive orthant R^{n+1}_+ can be expressed as $\{\Sigma^n_{i=0} k_i \vec{e}^{n+1}_{i+1} \mid k_i > 0\}$. It follows that

$$co(\tilde{C}^n_{\alpha,\beta}) = co(\tilde{T}_{\alpha,\beta}(\tilde{S})) = T_{\alpha,\beta}(co(\tilde{S})) = T_{\alpha,\beta}(R^{n+1}_+)$$

$$= \{\sum^n_{i=0} k_i T_{\alpha,\beta}(\vec{e}^{n+1}_{i+1}) \mid k_i > 0\}$$

$$= \{\sum^n_{i=0} k_i (q_{i0}, q_{i1}, \ldots, q_{in}) \mid k_i > 0\}$$

where $(q_{i0}, q_{i1}, \ldots, q_{in})$ is the ith row ($0 \le i \le n$) of matrix $Q_{\alpha,\beta}$. This gives us another characterization of $co(\tilde{C}^n_{\alpha,\beta})$ and $co(R\tilde{C}^n_{\alpha,\beta})$.

__Proposition 3.1.4.__ For any fixed $\alpha < \beta$ and n, $co(\tilde{C}^n_{\alpha,\beta})$ and $co(R\tilde{C}^n_{\alpha,\beta})$ are equal to the interior of the convex cone

generated by the vectors $\lambda_{j=0}^{n} \ q_{ij} e_{j+1}^{n+1}$, R^{n+1} with origin as
vertex, where for each i, the q_{ij}, $j = 0, 1, \ldots, n$ are the
coefficients of the polynomial expansion $(x-\alpha)^{n-i}(x-\beta)^{i} =$
$\Sigma_{j=0}^{n} \ q_{ij} x^{n-j}$. []

For $M \subset R^{n+1}$ and $a \in R$, let M_a denote the subset of M
defined by $\{(a_0, a_1, \ldots, a_n) \in M \mid a_0 = a\}$. Note that $\tilde{C}_{\alpha,\beta}^{n}$,
$C_{\alpha,\beta}^{n}$, $R\tilde{C}_{\alpha,\beta}^{n}$ and $RC_{\alpha,\beta}^{n}$ are related by the following relations:

$$(\tilde{C}_{\alpha,\beta}^{n})_1 = \{1\} \times C_{\alpha,\beta}^{n}$$

$$(R\tilde{C}_{\alpha,\beta}^{n})_1 = \{1\} \times RC_{\alpha,\beta}^{n}$$

The next lemma shows the relation between their convex hulls.
This will enable us to answer the first question which we
raised in the beginning of this chapter.

<u>Lemma 3.1.5.</u> (i) $(co(\tilde{C}_{\alpha,\beta}^{n}))_1 = \{1\} \times co(C_{\alpha,\beta}^{n})$ and $(co(R\tilde{C}_{\alpha,\beta}^{n}))_1 =$
$\{1\} \times co(RC_{\alpha,\beta}^{n})$.

(ii) $co(C_{\alpha,\beta}^{n}) = co(RC_{\alpha,\beta}^{n})$.

<u>Proof</u>: (i) Every point from $(co(\tilde{C}_{\alpha,\beta}^{n}))_1$, is of the form

$$\underset{i}{\Sigma} k_i (b_{i0}, b_{i1}, \ldots, b_{in})$$

where $(b_{i0}, b_{i1}, \ldots, b_{in}) \in \tilde{C}_{\alpha,\beta}^{n}$ and the k_i's satisfy the
relations $k_i > 0$, $\Sigma_i k_i = 1$ and $\Sigma_i k_i b_{i0} = 1$. Since, for each
$(b_{i0}, b_{i1}, \ldots, b_{in}) \in \tilde{C}_{\alpha,\beta}^{n}$, $b_{i0} > 0$, it follows that

$\underset{i}{\Sigma} k_i (b_{i0}, b_{i1}, \ldots, b_{in}) = \{1\} \times \underset{i}{\Sigma} k_i b_{i0} (b_{i1}/b_{i0}, b_{i2}/b_{i0}, \ldots,$
$b_{in}/b_{i0}) \in \{1\} \times co(C_{\alpha,\beta}^{n})$.

The last inclusion follows from the definition of $co(C_{\alpha,\beta}^{n})$.

Conversely, if $(1, a_1, a_2, \ldots, a_n) \in \{1\} \times co(C_{\alpha,\beta}^n)$, then

$(1, a_1, a_2, \ldots, a_n) = \{1\} \times \{\Sigma_i k_i (b_{i1}, b_{i2}, \ldots, b_{in})\} =$

$\Sigma_i k_i (1, b_{i1}, \ldots, b_{in})$ where $(b_{i1}, b_{i2}, \ldots, b_{in}) \in C_{\alpha,\beta}^n$ and

$\Sigma_i k_i = 1$ with $k_i > 0$. Since for each i, $(1, b_{i1}, b_{i2}, \ldots, b_{in}) \in$

$\widetilde{C}_{\alpha,\beta}^n$; this implies that $(1, a_1, \ldots, a_n) \in (co(\widetilde{C}_{\alpha,\beta}^n))_1$. Therefore,

$(co(\widetilde{C}_{\alpha,\beta}^n))_1 = \{1\} \times co(C_{\alpha,\beta}^n)$. The same proof holds for

$(co(R\widetilde{C}_{\alpha,\beta}^n))_1 = \{1\} \times co(RC_{\alpha,\beta}^n)$.

(ii) Since $co(\widetilde{C}_{\alpha,\beta}^n) = co(R\widetilde{C}_{\alpha,\beta}^n)$, by (i), it follows that

$\{1\} \times co(C_{\alpha,\beta}^n) = (co(\widetilde{C}_{\alpha,\beta}^n))_1 = (co(R\widetilde{C}_{\alpha,\beta}^n))_1 = \{1\} \times co(RC_{\alpha,\beta}^n)$.

This implies that $co(C_{\alpha,\beta}^n) = co(RC_{\alpha,\beta}^n)$. $\square$

Combining the previous lemma and proposition 3.1.3, we

obtain the description of the convex hulls of $C_{\alpha,\beta}^n$ and $RC_{\alpha,\beta}^n$.

<u>Proposition 3.1.6.</u> $Co(C_{\alpha,\beta}^n)$ and $co(RC_{\alpha,\beta}^n)$ are both equal to

$\{(a_1, a_2, \ldots, a_n) \in R^n \mid (-1)^j \binom{n}{j} \alpha^j \beta^{n-j} + \sum_{i=1}^n [(-1)^j \sum_{k=0}^j \binom{n-i}{j-k}$

$\binom{i}{k} \alpha^{j-k} \beta^{n-i-j+k}] a_i > 0, \ j = 0, 1, 2, \ldots, n \}.$

<u>Proof</u>: Apply proposition 3.1.3 with $a_0 = 1$. $\square$

Consider the matrix $Q_{\alpha,\beta} = (q_{ij})$ associated with $T_{\alpha,\beta}$.

Recall that q_{i0} equals one for each i, $0 \le i \le n$. From

proposition 3.1.4, it follows that

$\{1\} \times co(C_{\alpha,\beta}^n) = (co(C_{\alpha,\beta}^n))_1$

$= \{ \sum_{i=0}^n k_i (1, q_{i1}, q_{i2}, \ldots, q_{in}) \mid k_i > 0 \}_1$

$= \{1\} \times \{ \sum_{i=0}^n k_i (q_{i1}, q_{i2}, \ldots, q_{in}) \mid k_i > 0, \ \Sigma_i k_i = 1 \}.$

Hence,

$$\text{co}(C^n_{\alpha,\beta}) = \{\Sigma^n_{i=0}k_i(q_{i1},q_{i2},\ldots,q_{in}) \mid k_i > 0,\ \Sigma_i k_i = 1\}.$$

Proposition 3.1.7. $\text{co}(C^n_{\alpha,\beta})$ and $\text{co}(RC^n_{\alpha,\beta})$ are both equal to the open simplex generated by the n+1 points $(q_{i1},q_{i2},\ldots,q_{in}) \in R^n$, $i = 0,1,2,\ldots,n$ where q_{ij} are the coefficients of the polynomials $(x-\alpha)^{n-i}(x-\beta)^i = x^n + \Sigma^n_{j=1}q_{ij}x^{n-j}$; explicitly,

$$q_{ij} = (-1)^j\Sigma^j_{k=0}\binom{n-i}{j-k}\binom{i}{k}\alpha^{j-k}\beta^k. \qquad \Box$$

In particular, if $\alpha = -1$ and $\beta = 1$, then one obtains the following conclusions:

Corollary 3.1.8. $\text{co}(C^n_{-1,1})$ and $\text{co}(RC^n_{-1,1})$ are both equal to the open simplex generated by the n+1 points $(q_{i1},q_{i2},\ldots,q_{in}) \in R^n$, $i = 0,1,2,\ldots,n$ where q_{ij} are the coefficients of the polynomials $(x+1)^{n-i}(x-1)^i = x^n + \Sigma^n_{j=1}q_{ij}x^{n-j}$ explicitly,

$$q_{ij} = \sum^j_{k=0}(-1)^k\binom{n-i}{j-k}\binom{i}{k}.$$

Example: Suppose $n = 2$. Then, $\text{co}(C^2_{-1,1})$ is the shaded region in the figure 3.1, and is bounded by the following three lines:

$$1 + a_1 + a_2 = 0, \quad 1 - a_2 = 0, \quad 1 - a_1 + a_2 = 0$$

In this case, the convex hull of $C^2_{-1,1}$ is actually equal to $C^2_{-1,1}$ itself. Since

$$Q_{-1,1} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

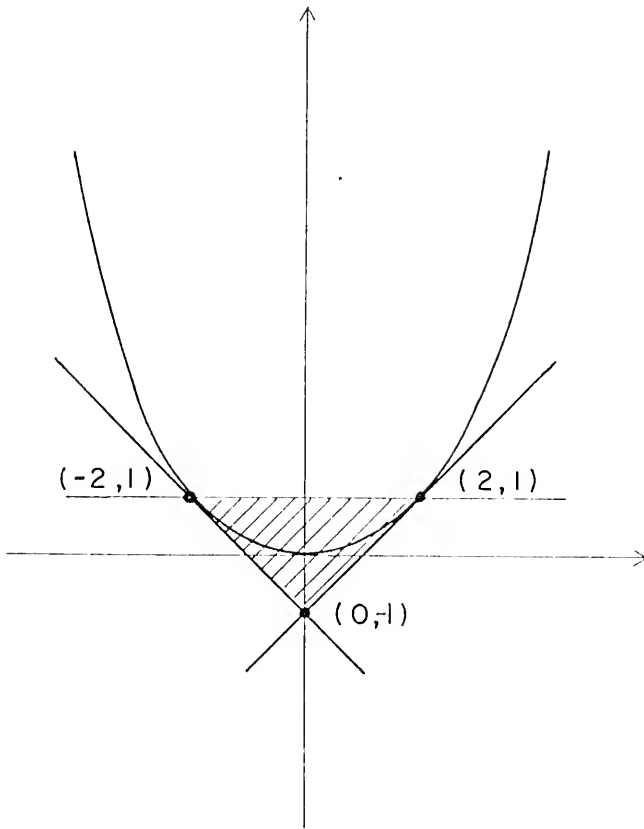Figure 3.1

by corollary 3.1.8, this implies that $co(C^2_{-1,1})$ is the triangle in R^2 having the points $(2,1)$, $(0,-1)$ ad $(-2,1)$ as vertices.

In fact, one may generalize proposition 3.1.7 to the following theorem.

Theorem 3.1.9. Given $\alpha < \beta$, let $I_{\alpha,\beta}$ be the open interval (α,β). Let $\mathcal{D}_{\alpha,\beta}$ be an open disk on the complex plane with α and β as the end points of a diameter, i.e., $\{z \mid |z-(\beta+\alpha)/2| < (\beta-\alpha)/2\}$. Let Γ be a set of complex numbers such that $I_{\alpha,\beta} \subseteq \Gamma \subseteq \mathcal{D}_{\alpha,\beta}$. Let $(\Gamma) = \{(a_1,a_2,\ldots,a_n) \in R^n \mid x^n + \Sigma^n_{j=0} a_j x^{n-j} = 0$ implies $x \in \Gamma\}$. Then, $co(\pi(\Gamma)) = co(C^n_{\alpha,\beta}) = co(RC^n_{\alpha,\beta})$.

Proof: The hypothesis on Γ implies that

$$RC^n_{\alpha,\beta} \subseteq \pi(\Gamma) \subseteq C^n_{\alpha,\beta}$$

Consequently,

$$co(RC^n_{\alpha,\beta}) \subseteq co(\pi(\Gamma)) \subseteq co(C^n_{\alpha,\beta}))$$

By the relation $co(RC^n_{\alpha,\beta}) = co(C^n_{\alpha,\beta})$, one concludes that

$$co(\pi(\Gamma)) = co(C^n_{\alpha,\beta}) = co(C^n_{\alpha,\beta}). \qquad []$$

3.2 $C^n_{\alpha,\beta}$-stabilizable Hyperplanes

Now, we consider the problem of characterizing hyperplanes which are stabilizable with respect to a given circle in the complex plane.

In the following, for any hyperplane $H = \{(a_1, a_2, \ldots, a_n)$
$\in R^n | A_0 + \sum_{i=1}^{n} A_i a_i = 0\}$, we denote by $\tilde{H}$ the set $\{(a_0, a_1, \ldots,$
$a_n) \in R^{n+1} | \sum_{i=1}^{n} A_i a_i = 0\}$. Note that $\tilde{H}$ is a hyperplane in R^{n+1}.

<u>Lemma 3.2.1.</u> Suppose H is a hyperplane in R^n. Then, for
any fixed $\alpha < \beta$, the property $H \cap C_{\alpha,\beta}^n \neq \phi$ is equivalent to
$\tilde{H} \cap \tilde{C}_{\alpha,\beta}^n \neq \phi$.

Proof: $(\Rightarrow)$. If $(a_1, a_2, \ldots, a_n) \in H \cap C_{\alpha,\beta}^n$, then by the
definitions of $\tilde{H}$ and $C_{\alpha,\beta}^n$ respectively, $(1, a_1, \ldots, a_n) \in \tilde{H}$ and
$(1, a_1, \ldots, a_n) \in \tilde{C}_{\alpha,\beta}^n$. Hence, $(1, a_1, a_2, \ldots, a_n) \in \tilde{H} \cap \tilde{C}_{\alpha,\beta}^n$.
$(\Leftarrow)$. If $(a_0, a_1, \ldots, a_n) \in \tilde{H} \cap \tilde{C}_{\alpha,\beta}^n$, then $(a_1/a_0, a_2/a_0, \ldots,$
$a_n/a_0) \in H \cap C_{\alpha,\beta}^n$. $\square$

Using this lemma and proposition 3.1.4, we obtain the
following criterion for determining the stabilizability of a
hyperplane with respect to a given circle.

<u>Theorem 3.2.2.</u> Given $\alpha < \beta$, let $Q_{\alpha,\beta}$ be the matrix associated
with the transformation $T_{\alpha,\beta}$ (see (3.8)). Suppose $H = \{(a_1, a_2,$
$\ldots, a_n) \in R^n | A_0 + \sum_{j=1}^{n} A_j a_j = 0\}$ is a hyperplane in R^n. Let
$\bar{A}_i$, $i = 0, 1, \ldots, n$, be the numbers obtaind from $(\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_n)^T =$
$Q_{\alpha,\beta}(A_0, A_1, \ldots, A_n)$, explicitly, $\bar{A}_i = \sum_{j=0}^{n} [(-1)^j \sum_{k=0}^{j} \binom{n-i}{j-k} \binom{i}{k} \alpha^{j-k}$
$\beta^k] A_j$. Then, the following statements are equivalent.

(1) $H \cap C_{\alpha,\beta}^n \neq \phi$,

(2) $H \cap RC_{\alpha,\beta}^n \neq \phi$,

(3) At least two of the numbers $\bar{A}_i$, $i = 0, 1, 2, \ldots, n$, are
of strictly opposite signs.

Proof: (1) $\Leftrightarrow$ (3). The proof follows from a sequence of equivalences below. First, observe that, by the last lemma, $H \cap C^n_{\alpha,\beta} \neq \phi$ is equivalent to $\tilde{H} \cap \tilde{C}^n_{\alpha,\beta} \neq \phi$. Since $\tilde{C}^n_{\alpha,\beta}$ is an open connected set in R^{n+1}, the property $\tilde{H} \cap \tilde{C}^n_{\alpha,\beta} \neq \phi$ holds if and only if $\tilde{H} \cap co(\tilde{C}^n_{\alpha,\beta}) \neq \phi$. Recall that $co(\tilde{C}^n_{\alpha,\beta})$ is equal to the open cone generated by the row vectors $(q_{i0}, q_{i1}, \ldots, q_{in})$, $i = 0,1,2,\ldots,n$ of the matrix $Q_{\alpha,\beta}$. Hence, $\tilde{H} \cap co(\tilde{C}^n_{\alpha,\beta}) \neq \phi$ means that there exist parameters $k_i > 0$, $i = 0,1,\ldots,n$, such that $\sum_{j=0}^{n} [\sum_{i=0}^{n} k_i q_{ij}] \tilde{e}_{j+1}^{n+1} \in \tilde{H}$ or equivalently,

$$\sum_{j=0}^{n} A_j (\sum_{i=0}^{n} k_i q_{ij}) = \sum_{i=0}^{n} (\sum_{j=0}^{n} q_{ij} A_j) k_i = 0$$

for some $k_i > 0$, $i = 0,1,2,\ldots,n$. By lemma 2.2.1, this is true if and only if at least two of the numbers $\bar{A}_i = \sum_{j=0}^{n} q_{ij} A_j$, $i = 0,1,\ldots,n$ are of strictly opposite signs.

(1) $\Leftrightarrow$ (2). Since $C^n_{\alpha,\beta}$ and $RC^n_{\alpha,\beta}$ are open connected sets, the property $H \cap C^n_{\alpha,\beta} \neq \phi$ holds if and only if $H \cap co(C^n_{\alpha,\beta}) \neq \phi$. Also, $H \cap RC^n_{\alpha,\beta} \neq \phi$ if and only if $H \cap co(RC^n_{\alpha,\beta}) \neq \phi$. Recall that $co(C^n_{\alpha,\beta}) = co(RC^n_{\alpha,\beta})$. Hence, $H \cap C^n_{\alpha,\beta} \neq \phi$ if and only if $H \cap RC^n_{\alpha,\beta} \neq \phi$. $\square$

CHAPTER IV
STABILIZABILITY OF A SYSTEM WITH DELAY

In this chapter we wish to illustrate the applicability of the "convex hull technique" which was used in the last two chapters, to the study of the stabilizability of the following delayed control system.

$(\mathcal{D})$ $\dot{x}(t) = \tilde{a}x(t) + \tilde{b}x(t-1) + u$, $y = \tilde{c}x(t) + \tilde{d}x(t-1)$, $u = ky$

where $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, k are real numbers.

Our purpose is to determine the existence of a real number k such that the corresponding characteristic equation of $(\mathcal{D})$

$$\lambda + (\tilde{a} + k\tilde{c}) + (\tilde{b} + k\tilde{d})e^{-\lambda} = 0 \tag{4.1}$$

has only roots in the left half complex plane. Note that with $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ fixed and $\tilde{c}^2 + \tilde{d}^2 \neq 0$, the set of all the coefficients of (4.1) which arise from all possible choices of $k \in R$ forms a straight line

$$H = \{(a,b) \mid a\tilde{d} - b\tilde{c} - (\tilde{a}\tilde{d} + \tilde{b}\tilde{c}) = 0\} \tag{4.2}$$

Now, consider an exponential polynomial equation of the form

$$\lambda + a + be^{-\lambda} = 0 \tag{4.3}$$

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Let $\mathcal{D}_\alpha$ denote the set $\{(a,b) \in R^2 \mid \lambda + a + be^{-\lambda} = 0 \text{ implies}$
$Re\lambda < \alpha\}$. A hyperplane H is said to be $\mathcal{D}_\alpha$-stabilizable if
$H \cap \mathcal{D}_\alpha \neq \phi$. In particular, when $\alpha = 0$ we call $\mathcal{D}_0$ the domain
of stability of (4.3). Thus, a system $(\mathcal{D})$ is stabilizable
if and only if $H \cap \mathcal{D}_0 \neq \phi$. As in the previous chapters, we
will study two main problems related to $\mathcal{D}_\alpha$: (i) the convex
hull of $\mathcal{D}_\alpha$ and (ii) the characterization of hyperplanes
(straight lines) which intersect $\mathcal{D}_\alpha$.

4.1 Convex Hull of $\mathcal{D}_0$

It is known that for any $(a,b) \in R^2$, equation (4.3) has
only a finite number of roots with positive real parts
(J. Hale [1], Lemma 20.1). This kind of roots will be called
unstable roots in the following. In terms of the D-decomposi-
tion method by Yu Naimark [1] one may partition the coefficient
space R^2 into infinitely many regions. We indicate in
figure 4.1 these regions and the number of unstable roots of
the equation when the coefficients belong to each region.
The boundary of these regions is the curve $a = \dfrac{-y \cos y}{\sin y}$
$b = \dfrac{y}{\sin y}$ where $n\pi < y < (n+1)\pi$ with $n = 0,1,2,\ldots$. This
figure actually suggests most of the techniques used in the
proofs of the theorems in this chapter.

Hayes [1] proved the following analytic characterization
of $\mathcal{D}_0$.

Theorem 4.1.1 [Hayes] All the roots of $\lambda + a + be^{-\lambda} = 0$,
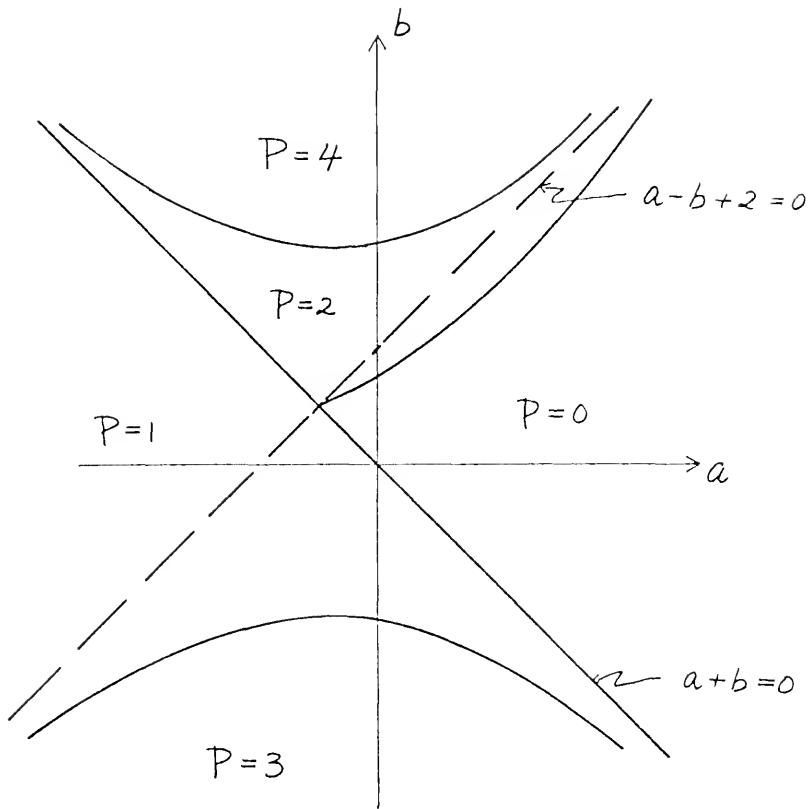where a and b are real, have negative real parts if and only if

    (i) $a + 1 > 0$

Figure 4.1

(ii) $a + b > 0$

(iii) $\sqrt{a^2+r^2} > b$ where if $a \neq 0$, r is the root of the
equation $r = -a \tan r$ in $(0,\pi)$ and if $a = 0$,
$r = \pi/2$. $\square$

Hence, $\mathcal{D}_0$ is the set of all points (a,b) which satisfy con-
ditions (i), (ii) and (iii) of the Hayes theorem. By the
continuity of the roots with respect to the coefficients of
exponential polynomials, one obtains that $\mathcal{D}_0$ is open in R^2.
Also,

Lemma 4.1.2. $\mathcal{D}_0$ is connected.

Proof: It suffices to show that for any pair of points
(a_1,b_1) and (a_2,b_2) form $\mathcal{D}_0$ there is a path $P \subset \mathcal{D}_0$ which
connects them.

Consider the half line $L = \{(a,1) \,|\, a > -1\}$. We assert
that $L \subset \mathcal{D}_0$. Since $(a,1) \in L$ implies $a + 1 > 0$, the condi-
tions (i) and (ii) of theorem 4.1.1 are satisfied. For
condition (iii), suppose first that $a \neq 0$. Then, by the
relations $r = -a \tan r$ and $0 < r < \pi$, one derives

$$\sqrt{a^2+r^2} = \sqrt{r^2\cot^2 r + r^2} = \frac{r}{\sin r} > 1 = b$$

If $a = 0$, then

$$\sqrt{a^2+r^2} = \sqrt{0+(\tfrac{\pi}{2})^2} > 1 = b$$

In both cases (iii) holds. Hence, $L \subset \mathcal{D}_0$. Now, for any
fixed (a_1,b_1) and (a_2,b_2), we construct the path P as
$P = P_1 \cup P_2 \cup P_3$, where

$$P_1 = \{(a_1,b) \mid b = 1+t(b_1-1), \ 0 \le t < 1\}$$

$$P_2 = \{(a,1) \mid a = a_1+t(a_2-a_1), \ 0 \le t \le 1\} \subset L \subset \mathcal{D}_0$$

$$P_3 = \{(a_2,b) \mid b = 1+t(b_2-1), \ 0 \le t \le 1\}$$

Note that if $(a_0,b_0) \in \mathcal{D}_0$, then for any b with $-a_0 < b < b_0$, the point (a_0,b) satisfies (i), (ii), and (iii) of theorem 4.1.1. By this property one obtains $P_1 \subset \mathcal{D}_0$. Similarly, $P_3 \subset \mathcal{D}_0$. Hence, $P \subset \mathcal{D}_0$. ☐

We remark that in general the stability domain of an exponential polynomial is not necessarily connected. For example, the stability domain of $\ddot{x}(t) + a\dot{x}(t-1) + bx(t) = 0$ is a disconnected open set in R^2 bounded by the straight lines $b = (-1)^{k+1} \frac{2k+1}{2}\pi a + [\frac{2k+1}{2}\pi]^2$, $k = 0,1,2,\ldots,n$. (See figure 4.2.) This suggests an interesting problem (not treated here): "What types of exponential polynomials have connected stability domains?"

In the following we are going to show that

$$co(\mathcal{D}_0) = \{(a,b) \in R^2 \mid a+1>0, \ a+b>0, \ a-b+2>0\} \tag{4.4}$$

The right hand side will be denoted by M in this chapter. This is a polygonal convex cone in R^2. To prove (4.4), since $co(\mathcal{D}_0)$ and M are open convex sets, it suffices to show that their closures are equal, namely, $\overline{co(\mathcal{D}_0)} = \overline{M}$. First, we need the following lemmas.

Lemma 4.1.3. Let $\mathcal{D}_0$ and M be defined as above. Then $\mathcal{D}_0 \subset M$.
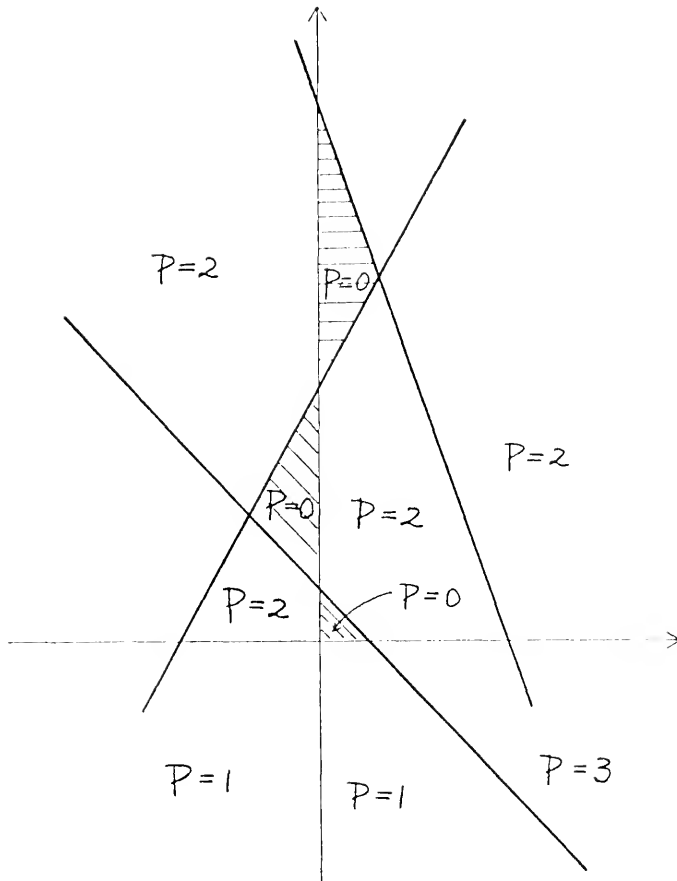
P=2

P=0

P=2

P=0 P=2

P=2

P=2 P=0

P=1 P=1

P=3

Figure 4.2

Proof: Observe that, by the Hayes theorem, any point (a,b) chosen from $\mathcal{V}_0$ satisfies $a + 1 > 0$ and $a + b > 0$. Therefore, one needs only to show that conditions (i), (ii) and (iii) of Hayes theorem imply $a - b + 2 > 0$.

Consider $a + 2 - \sqrt{a^2 + r^2}$ where $r = a \tan r$ with $0 < r < \pi$ when $a \neq 0$, and $r = \frac{\pi}{2}$ if $a = 0$. Suppose $a \neq 0$. Then, by the substitution of $a = r \cot r$, one obtains

$$a + 2 - \sqrt{a^2 + r^2} = -r \cot r + 2 - \sqrt{r^2 \cot^2 r + r^2}$$

$$= \frac{2(1 + \cos r)}{\sin r} \left(\tan \frac{r}{2} - \frac{r}{2}\right) > 0$$

The last inequality holds, since $\frac{1 + \cos r}{\sin r}$ as well as $\tan \frac{r}{2} - \frac{r}{2}$ are positive for all $0 < r < \pi$.

Suppose now $a = 0$, in which case $r = \pi/2$. Then, one has

$$a + 2 - \sqrt{a^2 + r^2} = 2 - \sqrt{\frac{\pi}{2}} > 0$$

In both cases, the inequality $a + 2 - \sqrt{a^2 + r^2} > 0$ holds. By (iii), this implies that $a + 2 - b > 0$. The proof is completed. $\square$

Lemma 4.1.4. The point $\theta = (-1,1)$ belongs to $\overline{co(\mathcal{V}_0)}$.

Proof: Let L be the half line defined in lemma 4.1.2. Recall that $L_1 \subset \mathcal{V}_0$. This lemma follows readily from the fact that for any $\varepsilon > 0$,

$$(-1 + \frac{\varepsilon}{2}, 1) \in L \cap N_\varepsilon(\theta) \qquad \square$$

For the convenience of the proof of the next lemma, we adopt the following equivalent representation of M;

$$M = \{(a,b) \mid a+1 > 0, \ b = ka + (k+1) \quad \text{for some } -1 < k < 1\} \qquad (4.5)$$

Lemma 4.1.5. $M \subseteq \overline{co(\mathcal{D}_0)}$.

Proof: Let (a_0, b_0) be an arbitrary point chosen from M.
Then, by (4.5) $b_0 = k_0 a_0 + (k_0+1)$ for some $-1 < k_0 < 1$. We
claim that for this fixed k_0 there exists an $\alpha > \max\{0, a_0\}$
such that the half line L_α defined as

$$L_\alpha = \{(a,b) \mid a > \alpha, \ b = k_0 a + (k_0+1)\}$$

lies in $\mathcal{D}_0$. Observe that if $\alpha > \max\{0, a_0\}$, then for any
$(a,b) \in L_\alpha$, one has $a + 1 > 0$; moreover,

$$a + b = a + k_0 a + (k_0+1) = (k_0+1)(a+1) > 0$$

since $-1 < k_0 < 1$ and $a_0 + 1 > 0$. Therefore, conditions (i)
and (ii) of theorem 4.1.1 are satisfied for all $(a,b) \in L_\alpha$,
α being any number greater than $\max\{0, a_0\}$.

For condition (iii), it suffices to show that for som
$\alpha > \max\{0, a_0\}$, one has $a^2 + r^2 - b^2 > 0$ for all $(a,b) \in L_\alpha$,
where $r = -a \tan r$ and $0 < r < \pi$. Since $a = -r \cot r$ and
$0 < r < \pi$, it follows that

$$a^2 + r^2 - b^2 = (-r \cot r)^2 + r^2 - [-k_0 r \cot r + (k_0+1)]^2$$
$$= r^2 \csc^2 r - [k_0 r \cot r - (k_0+1)]^2$$
$$= r^2 \csc^2 r \{1 - [k_0 \cos r - (k_0+1) \frac{\sin r}{r}]^2\}$$

Note that $1 - [k_0 \cos r - (k_0+1) \frac{\sin r}{r}]^2$ is continuous in a
neighborhood of $r = \pi$; moreover, one has

$$1 - [k_0\cos \pi - (k_0+1)\frac{\sin \pi}{\pi}]^2 = 1 - k_0^2 > 0$$

Hence, there exists an ε, with $0 < \varepsilon < \pi/2$, such that for all r satisfying $\pi - \varepsilon < r < \pi$,

$$a^2+r^2-b^2 = r^2\csc^2 r\{1 - [k_0\cos r - (k_0+1)\frac{\sin r}{r}]^2\}>0 \quad (4.6)$$

Now, let $\alpha = 2 \max\{0,\bar{a}, -(\pi-\varepsilon)\cot(\pi-\varepsilon)\}$. Note that $-r \cot r$ is an increasing function of r in the interval $0 < r < \pi$. Hence, for every $a > \alpha$, the corresponding r obtained from $-r \cot r = a$ satisfies the inequalities $\pi - \varepsilon < r < \pi$. Consequently, (4.6) holds for all $(a,b) \in L_\alpha$. This completes the proof of the claim.

Now, by this claim, one may represent (a_0,b_0) as a convex combination of the point $(-1,1)$ and some $(a,b) \in L_\alpha \subset \mathcal{D}_0$, i.e.,

$$(a_0,b_0) = \frac{a-a_0}{1+a} (-1,1) + (1 - \frac{a-a_0}{1+a})(a,b).$$

Therefore, $(a_0,b_0) \in \overline{co(\mathcal{D}_0)}$. Hence, one concludes that $M \subseteq \overline{co(\mathcal{D}_0)}$. □

Proposition 4.1.6. $co(\mathcal{D}_0) = \{(a,b) \in R^2 \mid a+1>0, a+b>0, a-b+2>0\}$.

Proof: By lemma 4.1.3 and the convexity of M, one obtains $\overline{co(\mathcal{D}_0)} \subseteq \overline{co(M)} = \bar{M}$. On the other hand, lemma 4.1.5 implies that $\bar{M} \subseteq \overline{co(\mathcal{D}_0)}$. Therefore, $\bar{M} = \overline{co(\mathcal{D}_0)}$. Since for any m dimensional convex set $Y \in R^m$, interior $(\bar{Y})$ = interior (Y) (see e.g., Rockafellar [1] p. 46), also since $co(\mathcal{D}_0)$ and

M are open convex sets in R^2, one concludes that $co(\mathcal{D}_0) = M = \{(a,b)|a + 1 > 0, a + b > 0, a - b + 2 > 0\}$. ☐

4.2 $\mathcal{D}_\alpha$-Stabilizable Hyperplanes

Now, let H be a hyperplane in R^2 defined by $Aa + Bb + C = 0$. Since $\mathcal{D}_0$ is an open connected set, it follows that $H \cap \mathcal{D}_0 \neq \phi$ if and only if $H \cap co(\mathcal{D}_0) \neq \phi$. Observe that, equivalently, the open convex cone $co(\mathcal{D}_0)$ can be expressed as:

$$co(\mathcal{D}_0) = \{(-1+k_1+k_2, \ 1+k_1-k_2)|k_1 > 0, \ k_2 > 0\}$$

Hence, $H \cap co(\mathcal{D}_0) \neq \phi$ means that there exist $k_1 > 0$ and $k_2 > 0$ such that $(-1+k_1+k_2, \ 1+k_1-k_2) \in H$ or equivalently, $A(-1+k_1+k_2) + B(1+k_1-k_2) + C = (A+B)k_1 + (A-B)k_2 + (B-A+C) = 0$ from some $k_1 > 0$ and $k_2 > 0$. By lemma 2.2.1, this is true if and only if at least two of the numbers $A + B$, $A - B$, $B - A + C$ are of strictly opposite signs.

Summarizing the analysis above, we obtain the following theorem.

Theorem 4.2.1. Suppose H is a hyperplane in R^2 defined by $Aa + Bb + C = 0$. Then, $H \cap \mathcal{D}_0 \neq \phi$ if and only if at least two of the numbers $A - B$, $A + B$, $B - A + C$ are of strictly opposite signs. ☐

Expressed in terms of the original parameters $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, this theorem can be restated as:

Theorem 4.2.2. The system (D) is $\mathcal{D}_0$-stabilizable if and only if at least two of the numbers $\tilde{d}-\tilde{c}$, $\tilde{d}+\tilde{c}$, $-\tilde{c}-\tilde{d}-\tilde{a}\tilde{d}-\tilde{b}\tilde{c}$ are of strictly opposite signs.

Proof: Recall from (4.2) that the hyperplane corresponding to $(\mathcal{D})$ is defined by $\tilde{d}a - \tilde{c}b - (\tilde{a}\tilde{d}+\tilde{b}\tilde{c}) = 0$. Hence, the A, B, C in the previous theorem are $\tilde{d}$, $-\tilde{c}$ and $-(\tilde{a}\tilde{d}+\tilde{b}\tilde{c})$, respectively. $\square$

One of the most interesting questions in the stabilization of a delayed control system is the following: "Can a system with delay be stabilized by a feedback without delay?" In general, this cannot be done as the example at the end of this chapter shows. However, for a system like $(\mathcal{D})$, the theorem above tells us that $(\mathcal{D})$ may indeed be stabilized by a feedback without delay. In fact, under this kind of feedback, one has $\tilde{d} = 0$ and consequently, $\tilde{d} - \tilde{c} = -\tilde{c}$ and $\tilde{d} + \tilde{c} = \tilde{c}$ are of strictly opposite signs.

Given $\alpha \in R$, recall that $\mathcal{D}_\alpha$ is the set of all coefficients (a,b) such that the real parts of all the roots of $\lambda + a + be^{-\lambda} = 0$ are smaller than α. Consider a substitution of λ by $\eta + \alpha$ in equation (4.3).

$$\lambda + a + be^{-\lambda} = \eta + (a+\alpha) + be^{-\alpha}e^{-\eta}$$

The substitution $\lambda = \eta + \alpha$ induces an affine transformation T_α of the coefficient space R^2 of equation (4.3). This transformation is defined by

$$T_\alpha(a,b) = (x,y) = (a,b) \begin{pmatrix} 1 & 0 \\ 0 & e^{-\alpha} \end{pmatrix} + (\alpha, 0)$$

Since $\det \begin{pmatrix} 1 & 0 \\ 0 & e^{-\alpha} \end{pmatrix} = e^{-\alpha} \neq 0$, T_α is invertible. Its inverse is defined by

$$T_\alpha^{-1}(x,y) = (a,b) = (x,y) \begin{pmatrix} 1 & 0 \\ 0 & e^\alpha \end{pmatrix} + (-\alpha,0)$$

Note that $\lambda = \alpha + \eta$ implies that the condition $Re(\lambda) < \alpha$ is equivalent to $Re(\eta) < 0$. Hence, T_α maps $\mathcal{D}_\alpha$ onto $\mathcal{D}_0$, i.e., $T_\alpha(\mathcal{D}_\alpha) = \mathcal{D}_0$.

Suppose that H is a hyperplane in R^2 defined by $Aa + Bb + C = 0$. Then $H \cap \mathcal{D}_\alpha \neq \phi$ if and only if $\phi \neq T_\alpha(H \cap \mathcal{D}_\alpha) = T_\alpha(H) \cap T(\mathcal{D}_\alpha) = T_\alpha(H) \cap \mathcal{D}_0$.

Observe that

$$\begin{aligned}
T_\alpha(H) &= \{(x,y) \mid T_\alpha^{-1}(x,y) \in H\} \\
&= \{(x,y) \mid [(x,y)\begin{pmatrix} 1 & 0 \\ 0 & e^\alpha \end{pmatrix} + (-\alpha,0)]\begin{pmatrix} A \\ B \end{pmatrix} + C = 0\} \\
&= \{(x,y) \mid Ax + Be^\alpha y + C - A\alpha = 0\}
\end{aligned}$$

Therefore, by theorem 4.1.1 we have the following generalization.

<u>Theorem 4.2.3.</u> Suppose that H is the hyperplane in R^2 defined by $Aa + Bb + C = 0$. Then $H \cap \mathcal{D}_\alpha \neq \phi$ if and only if at least two of the numbers $A - Be^\alpha$, $A + Be^\alpha$ and $Be^\alpha + C - A - A\alpha$ are of strictly opposite signs. $\qquad\Box$

Recall that for polynomials of degree n, the number $\inf\{\alpha \mid H \cap S_\alpha^n \neq \phi\}$ is always bounded from below for any hyperplane H. Indeed, theorem 2.3.1 describes this property. However the next theorem shows that this is not the case for the $\mathcal{D}_\alpha$'s.

Theorem 4.2.4. Suppose H is the hyperplane in R^2 defined by $Aa + Bb + C = 0$. If $A = 0$, then $\inf\{\alpha | H \cap \mathcal{D}_\alpha \neq \phi\} = -\infty$. If $A = 0$, then

$$\inf\{\alpha | H \cap \mathcal{D}_\alpha \neq \phi\} = \inf\{\alpha \in R | (A-Be^\alpha)(A+Be^\alpha)(Be^\alpha+C-A-A\alpha) = 0\} \quad (4.7)$$

Proof: If $A = 0$, then $A + Be^\alpha = Be^\alpha$ and $A - Be^\alpha = -Be^\alpha$. These numbers are always of opposite signs for any $\alpha \in R$. Hence, $\inf\{\alpha | H \cap \mathcal{D}_\alpha \neq \phi\} = -\infty$. Suppose $A \neq 0$. Without loss of generality, assume $A > 0$. Let denote the number on the right hand side of (4.7). Note that, for any $\beta < \eta$, the numbers $A - Be^\beta$, $A + Be^\beta$ and $Be^\beta + C - A - A\beta$ all have the same sign as A. Hence, $\eta \leq \inf\{\alpha | H \cap \mathcal{D}_\alpha \neq \phi\}$. On the other hand, we have one of the following four cases:

(1) If η satisfies $A - Be^\eta = 0$, then there exists an $\varepsilon > 0$ such that $(A-Be^\alpha)A < 0$ and $(A+Be^\alpha)A > 0$ for all α such that $\eta < \alpha < \eta + \varepsilon$. Therefore

$$(A-Be^\alpha)(A+Be^\alpha) > 0 \qquad\qquad (4.8)$$

for all α such that $\eta < \alpha < \eta + \varepsilon$.

(2) Suppose that η satisfies $A + Be^\eta = 0$. Then, as in (1), one may find $\varepsilon > 0$ small enough such that (4.8) holds.

(3) Suppose η is a double root of $Be^\alpha + C - A - A\alpha = 0$. This implies that $Be^\eta - A = 0$. This case is then reduced to case (1).

(4) If η is a root of multiplicity one of $Be^\alpha + C - A - A\alpha = 0$, then one may find $\varepsilon > 0$ such that

$(Be^\alpha + C - A - A\alpha) \; A < 0$ and $(A - Be^\alpha) \; A > 0$ for

all α such that $\eta < \alpha < \eta + \epsilon$. Therefore,

$(Be^\alpha + C - A - A\alpha)(A - Be^\alpha) < 0$ for all α such that

$\eta < \alpha < \eta + \epsilon$. From the discussion above, one

concludes that $\min\{\alpha \,|\, H \cap D_\alpha \neq \phi\} \leq \eta$.

Hence, $\eta = \inf\{\alpha \,|\, H \cap D_\alpha \neq \phi\}$.   []

Remark: The stability domain of an arbitrary ixed type of

exponential polynomial $H(x, e^x)$ is one of the most interesting

subsets of the coefficient space of $H(x, e^x)$. Most of the

properties of the stability domains of polynomials are not

necessarily true for stability domains of exponential polyno-

mials. For example, the stability domain of $\lambda^2 + (a\lambda + b)e^{-\lambda} = 0$

is the set bounded by $b = 0$ and curve $a = y \sin y$, $b = y^2 \cos y$

where $0 \leq y \leq \frac{\pi}{2}$. (see El'sgol'ts [1]) Illustrated by figure

4.3, we see that this is a bounded convex set. Recall,

however, that the stability domain of any polynomial of

degree greater than or equal to three is an unbounded noncon-

vex set. In chapter one, we also mentioned that the stability

domain of any polynomial is connected. However, the stability

domain of the equation $\lambda^2 + a\lambda e^{-\lambda} + b = 0$ (see figure 4.2)

is disconnected.

It seems that a "nice" characterization of the stability

domains of exponential polynomials is essential for under-

standing these topological properties mentioned above, or,

even the stability domain itself. The most general charac-

terization available is a theorem proved by Pontryagin [1]

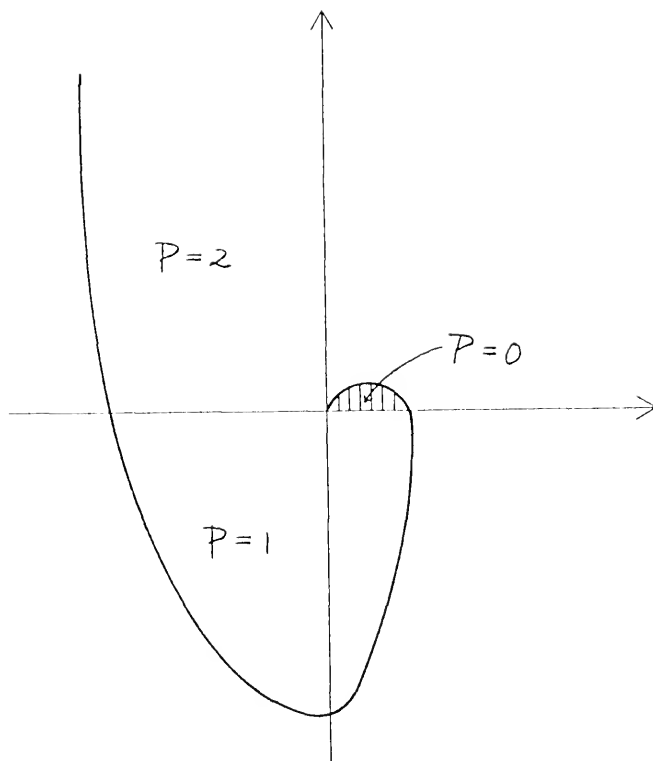(see next section). However, the difficulty in applying

Figure 4.3

Pontryagin's theorem arises from the fact that, generally, it is not easy to determine whether a transcendental equation has all the roots real or not. Bellman and Cooke's book [1] contains characterizations of stability domains for some special exponential polynomials. Much still remains to be done for a better understanding of the stability domain of general exponential polynomials.

4.3 An Example

In this section, we give an example of a control system with delay which can not be stabilized by feedbacks without delay. First, let us review a theorem proved by Pontryagin [1].

Definition: Let $h(z,w) = \sum_{m,n} a_{mn} z^m w^n$ with m,n nonnegative integers. A term $a_{rs} z^r w^s$ is called the principal term of $h(z,w)$ if $a_{rs} \neq 0$ and, if for each other term a_{mn} with $a_{mn} \neq 0$, we have either $r > m$, $s > n$ or $r = m$, $s > n$ or $r > m$, $s = n$.

Theorem 4.3.1. [Pontryagin] Let $H(z) = h(z,e^z)$, where $h(z,t)$ is polynomial with a principal term. The function $H(iy)$ is now separated into real and imaginary parts i.e., we set $H(iy) = F(y) + iG(y)$. If all the roots of the function $H(z)$ lie to the left hand side of the imaginary axis, then the zeros of the functions $F(y)$ and $G(y)$ are real, inter-lacing and

$$G'(y)F(y) - G(y)F'(y) > 0 \qquad (4.9)$$

for each $y \in R$. Moreover, in order that all the roots of the function lie to the left of the imaginary axis, it is sufficient that one of the following conditions be satisfied:

(1)   All the zeros of the functions $F(y)$ and $G(y)$ are real and interlacing and the inequality (4.9) is satisfied for at least one value of y;

(2)   All the zeros of the function $F(y)$ are real and for each zero $y = y_0$ of F, condition (4.9) is satisfied i.e., $F'(y_0)G(y_0) < 0$;

(3)   All the zeros of the function $G(y)$ are real and for each zero $y = y_0$ of G the inequality (4.9) is satisfied i.e., $G'(y_0)F(y_0) > 0$.

Lemma 4.3.2.   If the exponential polynomial

$$h(\lambda) = e^{\lambda}(\lambda^2 + p\lambda + q) + (r\lambda + m) = 0 \qquad (4.10)$$

has only stable roots, then $(q + m)(p + q + r) > 0$.

Proof: Observe that, by substituting iy for λ, one obtains $h(iy) = F(y) + iG(y)$ where

$$F(y) = (q - y^2)\cos y + m - py \sin y$$
$$G(y) = (q - y^2)\sin y + py \cos y + ry$$

From Pontryagin's Theorem and the hypothesis that $h(\lambda)$ has only stable roots, it follows that $G'(y)F(y) - G(y)F'(y) > 0$ for all $y \in R$. In particular, if $y = 0$, then $G'(0)F(0) - G(0)F'(0) = (q + m)(p + q + r) > 0$. Now, consider the following control system:

$$(E) \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad , \quad u = [f_1, f_2] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where a_i, f_i, $i = 1, 2$ are real numbers. Note that there is no delay term in feedback control. The characteristic equation of (E) is

$$\lambda^2 + \lambda(a_2 - f_2) + (a_1 - f_1) + e^{-\lambda}[\lambda - (a_2 - f_2 + 1 + 2a_1 - 2f_1)] = 0$$

Compare this equation with (4.10). Then

$$(q + m)(q + p + r) = [(a_1 - f_1) - (a_2 - f_2 + 1 + 2a_1 - 2f_1)][a_1 - f_1 +$$

$$a_2 - f_2 + 1] = - [(a_1 - f_1) + (a_2 - f_2) + 1]^2 \le 0$$

for any real numbers f_1 and f_2. Hence, the system (E) is always unstable for any feedback without delay.

CHAPTER V
CONCLUDING REMARKS

Consider a system $(\pi, K)^n$ with parameters in K. In chapter 2 and chapter 3 we discussed respectively the S_α-stabilizability and $C_{\alpha, \beta}$-stabilizability of a system $(\pi, K)^n$ for which $\chi \pi(K)$ forms a hyperplane. Suppose $\chi \pi(K)$ contains only one point. Then, $\chi \pi(K) \cap S_\alpha^n \neq \phi$ is equivalent to $\chi \pi(K) \subset S_\alpha^n$. One can use the well known Routh-Hurwitz criterion and the transformation T_α constructed in section 2.1 to determine if a point belongs to S_α^n. We now know how to characterize S_α-stabilizability of affine sets with dimension equal to n, n-1, and 0. For affine sets with dimensions between n-2 and 1 inclusive, no complete answer is known yet. Observe that in general, for an affine set of dimension lower than n-1, the fact that this affine set intersects the convex hull of certain set S does not always imply that the intersection of that affine set and S is nonempty. Hence, the "convex hull technique" gives less information for affine sets with dimension lower than n-1, since only necessary conditions are obtained. However, because of their simplicity when used on a computer, those necessary conditions can be used as a quick check to eliminate systems which are "obviously" not S_α-stabilizable. Skoog [1] obtained some partial results of S_0-stabilizability for the case of straight lines in terms of Nyquist

plot. In an inspiring paper, Anderson, Bose and Jury [1] showed that Output Feedback Stabilization problem can be handled by algorithms from decision algebra (see Jacobson [1]). This can be a very fruitful direction to pursue. It is also interesting to point out that the case of a hyperplane, which is the most complicated among all affine sets with dimension less than n-1 when one uses decision algebra, turns out to be the easiest for our approach, but the case of a line which is the most difficult for us is the easiest by decision algebra algorithm.

In the following, we will present a partial result and a computer aided algorithm concerning lines which intersect S_0^n. They should only be considered as an attempt in this direction.

Proposition 5.1.1. Suppose $f(x)$ and $g(x)$ are monic polynomials of degree n and n-1 respectively. Then, for any $\varepsilon > 0$, $\Lambda < 0$, there exists a $K \in R$ such that for every $k > K$, n-1 of the roots of $f(x) + kg(x) = 0$ are within the ε-neighborhoods of the zeros of $g(x)$ and the remaining root is smaller than Λ.

Proof: Let $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the n-1 roots of $g(x) = 0$ (counting the multiplicity). For each i, let C_i be a circle centered at λ_i with radius $r_i < \varepsilon$ such that the Interior(C_i) does not contain any $\lambda_j \neq \lambda_i$. Let $M_i^1 = \sup_{z \in C_i} |f(z)|$ and $M_i^2 = \inf_{z \in C_i} |g(z)|$. Choose $K' > 0$, such that $M_i^1 < M_i^2 K'$ for each i = 1, 2, ..., n-1. Hence, $|f(z)| < |kg(z)|$ for any $k > K'$,

$z \in C_i$ and $i = 1, 2, \ldots, n-1$. Therefore, by Rouché theorem, $f(x) + kg(x) = 0$ has as many zeros in Interior(C_i) as $g(x) = 0$ does for any $k > K'$ and $i = 1, 2, \ldots, n-1$. Now, consider $k = -f(x)/g(x)$. Since $(-f(x)/g(x))' = -1 + p(x)/g^2(x)$ with deg $p(x) < $ deg $g^2(x)$ there exist $\Lambda_1 \in R$ such that $-f(x)/g(x)$ is strictly decreasing in the interval $(-\infty, \Lambda_1)$. Also, since deg f $-$ deg $g = 1$, it follows that there exists some Λ_2 such that $k = -f(x)/g(x) > 0$ for all $x < \Lambda_2$ and $\lim_{x \to -\infty} (-f(x)/g(x)) = \infty$. Hence, $k = -f(x)/g(x)$ is invertible in $(-\infty, \alpha)$ with $\alpha = \min\{\Lambda_1, \Lambda_2, \Lambda\} - 1$. This implies that for every $k > -f(\alpha)/g(\alpha)$, the x obtained from $k = -f(x)/g(x)$ satisfies $x \le \alpha < \Lambda$, or equivalently, $f(x) + kg(x)$ has a zero smaller than Λ for every $k > -f(\alpha)/g(\alpha)$. Now the proposition follows readily if one chooses $K = \max\{K', -f(\alpha)/g(\alpha)\}$. $\square$

This proposition shows an interesting geometric property of S_0^n: Every straight line in R^n with a direction $(r_1, r_2, \ldots, r_n)$ where $(r_2/r_1, r_3/r_1, \ldots, r_n/r_1) \in S_0^{n-1}$, with $r_1 \neq 0$ intersects S_0^n. Furthermore, an infinite portion of the line is contained in S_0^n.

For straight lines which do not have such kind of "direction" we propose the following algorithm.

Consider $f(x) = x^n + \Sigma_{j=1}^n a_j x^{n-j}$ and $g(x) = \Sigma_{j=1}^n r_j x^{n-j}$. Let $f(x) + kg(x) = x^n + \Sigma_{j=1}^n (a_j + kr_j) x^{n-j} = p(x, k)$.

(i) Form the Hurwitz matrix $H(k)$ of $p(x, k)$ treated as a polynomial of x.

(ii) Solve the real roots of $\det(H(k)) = 0$. (For digital computer, this should not be a difficult task). Let the distinct real roots be $k_1 < k_2 < , \ldots, < k_m$.

(iii)   Choose arbitrary points $\tilde{k}_i \in (k_i, k_{i+1})$, $i = 0,1,2,\ldots,m$
satisfying $-\infty = k_0 < \tilde{k}_0 < k_1 < \tilde{k}_1 < k_2 < \tilde{k}_2 < \ldots < k_m < \tilde{k}_m < k_{m+1} = \infty$.

(iv)   Use Lienard-Chipart criterion to test each $f + \tilde{k}_i g$,
$i = 0,1,\ldots,m$.   If there exists a $0 \le i \le m$ such that
$f(x) + \tilde{k}_i g(x) = 0$ has only stable roots, then for all
$k \in (k_i, k_{i+1})$ the $f(x) + kg(x) = 0$ will have only stable roots.
Otherwise, the straight line $\{(a_1 + kr_1,\ldots,a_n + kr_n)\,|\,k \in R\}$
does not intersect S_0^n.

The proof of this algorithm follows from the next lemma.

Lemma 5.1.2.   Let $f(x)$ and $g(x)$ be two polynomials of degrees
n and ℓ respectively with $\ell < n$.   Let k_i, $i = 1,2,\ldots,m$ be
the distinct real roots of $\det(H(k)) = 0$ where $H(k)$ is the
Hurwitz matrix of $f(x) + kg(x)$.   Then, for each fixed i,
all the polynomial equations $f(x) + kg(x) = 0$ with $k \in (k_i, k_{i+1})$
will have the same number of unstable roots.

Proof: Let i be fixed.   For each $k \in (k_i, k_{i+1})$, let $p_j(k)$,
$j = 1,2,\ldots,n$ denote the roots of $f(x) + kg(x) = 0$.   Recall
that, by Orlando's formula (Gantmacher [1] vol. 2, p. 196),
$\det(H(k)) = (-1)^{n(n+1)/2} \prod_{j=1}^{n} \lambda_j(k) \cdot \prod_{r<w}(\lambda_r(k) + \lambda_w(k))$.   If
there exist k' and k'' both in (k_i, k_{i+1}) and $k' < k''$, such
that $f(x) + k'g(x)$ and $f(x) + k''g(x)$ have different number
of unstable roots, then by the continuity property of roots
with respect to the coefficients, there exists $\tilde{k}$ with
$k_i < k' < \tilde{k} < k'' < k_{i+1}$ such that the equation $f(x) + \tilde{k}g(x) = 0$
has either a pair of conjugate of purely imaginary roots or a

zero root. Hence, $\det(H(\tilde{k})) = 0$. This contradicts the fact
that between k_i and k_{i+1}, there is no real root of $\det(H(k)) = 0$.
Therefore, for every $k \in (k_i, k_{i+1})$, the polynomials $f + kg$
have the same number of unstable roots. ▯

By this lemma, one sees that it is enough to test only
one point from each interval (k_i, k_{i+1}) to determine whether
all the roots of $f(x) + kg(x) = 0$ are stable or not for any
$k \in (k_i, k_{i+1})$. The major advantage of this algorithm over the
conventional root locus method is that it involves only solving
real roots of an nth degree polynomial. This is a well devel-
oped area in numerical analysis. One may program that easily
on a digital computer. The other advantage is that it also
gives the numerical ranges of k for which $f + kg$ has only
stable roots. The major drawback of this algorithm is that
it is very sensitive to double real roots. A small round off
error may cause the loss of a double real roots of $\det(H(k)) = 0$.
In other words, one may lose one of the partition points k_i.
However, this can be prevented by predetermining the number
of distinct real roots by Sturm's theorem (see Gantmacher [1]
vol. 2, p. 175). Note this algorithm also works for any poly-
nomial of the form $\Sigma_{j=0}^{n} f_j(k) x^{n-j}$ when f_j are polynomials of k.

Example: Suppose $f(x) = x^3 + x^2 - x + 5$ and $g(x) = x^2 + 3x - 1$.
We wish to find the range of k such that $f + kg = 0$ has only
stable roots.

(i) $\det(H(k)) = 3(-k+5)(k+2)(k-1)$.

(ii) The three roots $k = 5$, $k = -2$, $k = 1$ separate the
real line into four intervals $(-\infty, -2)$, $(-2, 1)$, $(1, 5)$, $(5, \infty)$.

(iii)   Pick one point $\tilde{k}_i$ from each interval above and substitute them into $H(k)$. It is easy to check that the only $H(\tilde{k}_i)$ which satisfies Lienard-Chipart criterion is the one with $\tilde{k}_i \in (1,5)$.

Hence, one concludes that every polynomial equation $f + kg = 0$ with $k \in (1,5)$ has only stable roots.

For further research, this author feels that the study of the S_α-stabilizability of affine sets with dimension strictly lower than n-1 deserves an immediate attention. Besides the engineering application, this study also provides a better understanding of the set S_α^n itself. After this, the investigation of S_α-stabilizability of systems with multiple control would be the next goal. For nonlinear control systems, a problem, initially proposed by Letov [1], of determining the minimal number state measurements needed for stabilizing the system by a feedback control was considered by Casti and Letov [1].

In a practical situation, it is not enough just to know that the system is stabilizable by a constant gain feedback. The numerical scheme for finding a feedback gain matrix is also important. The papers by Miller, Cochran and Howze [1]; Luus [1], McBrinn and Roy [1], as well as Sirisena and Choi [1] represent some of the research in this direction.

APPENDIX I

Given $a \in R$ recall that

$$S_\alpha^n = \{(a_1, a_2, \ldots, a_n) \in R^n | \ x^n + \Sigma_{j=1}^n a_j x^{n-j} = 0 \ \text{implies}$$

$$Re(x) < \alpha\}$$

In this appendix we wish to show that S_α^n is connected. Since, in R^n, connectedness is equivalent to path connectedness, it suffices to show that for every a' = $(a_1', a_2', \ldots, a_n') \in S_\alpha^n$, a" = $(a_1'', a_2'', \ldots, a_n'') \in S_\alpha^n$ there exists a path p: $[0,1] \to R^n$ such that $p([0,1]) \subset S_\alpha^n$ with p(0) = a' and p(1) = a". Now consider a mapping π from C^n into C^n where C is the set of all complex numbers, such that for each $(x_1, x_2, \ldots, x_n) \in C^n$,

$$\pi(x_1, x_2, \ldots, x_n) = (a_1, a_2, \ldots, a_n)$$

where a_j's satisfy the relation $\{x | \ x^n + \Sigma_{j=1}^n a_j x^{n-j} = 0\}$ = $\{x_1, x_2, \ldots, x_n\}$. Note that π is continuous. Also, if $\{x_1, x_2, \ldots, x_n\}$ is symmetric with respect to the real axis i.e., $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ = $\{x_1, x_2, \ldots, x_n\}$ where $\bar{x}_j$ represents the conjugate of x_j, then $\pi(x_1, x_2, \ldots, x_n) \in R^n$.

Now, for the a' and a" given above, arrange the roots of $x^n + \Sigma_{j=1}^n a_j' x^{n-j} = 0$ and $x^n + \Sigma_{j=1}^n a_j'' x^{n-j} = 0$ such that

$$\{x \mid \sum_{j=1}^{n} a_j' x^{n-j}\} = \{x_1', x_2', \ldots, x_{2m}', x_{2m+1}', \ldots, x_n'\}$$

$$\{x \mid \sum_{j=1}^{n} a_j'' x^{n-j}\} = \{x_1'', x_2'', \ldots, x_{2r}'', x_{2r+1}'', \ldots, x_n''\}$$

where x_i', $i = 2m+1, \ldots, n$ and x_j'', $j = 2r+1, \ldots, n$ are real roots; x_i', $i = 1, 2, \ldots, 2m$, and x_j', $j = 1, 2 \ldots, 2r$ are complex roots such that $\bar{x}_{2j}' = x_{2i-1}'$, $i = 1, 2, \ldots, m$ and $\bar{x}_{2j}'' = x_{2j-1}''$, $j = 1, 2, \ldots, r$. Note that $\pi(x_1', x_2', \ldots, x_n') = (a_1', \ldots, a_n')$ and $\pi(x_1'', x_2'', \ldots, x_n'') = (a_1'', a_2'', \ldots, a_n'')$. Consider the path $\tilde{p}: [0,1] \to C^n$ defined by

$$\tilde{p}(t) = t(x_1', x_2', \ldots, x_n') + (1-t)(x_1'', x_2'', \ldots, x_n'')$$

Note for each $t \in [0,1]$ the set $\{tx_1' + (1-t)x_1'', tx_2' + (1-t)x_2'', \ldots, tx_n' + (1-t)x_n''\}$, is symmetric with respect to the real axis. Hence, $\pi(\tilde{p}([0,1])) \subset R^n$. Also, since $a' \in S_\alpha^n$ and $a'' \in S_\alpha^n$, by the definition of S_α^n, one has $Re(x_i') < \alpha$ and $Re(x_i'') < \alpha$ for every $i = 1, 2, \ldots, n$. It follows that for any $t \in [0,1]$ and for every $i = 1, 2, \ldots, n$,

$$Re(tx_i' + (1-t)x_i'') = t\, Re(x_i') + (1-t)\, Re(x_i'') < \alpha$$

Therefore, $\pi(\tilde{p}([0,1])) \subset S_\alpha^n$. Note that $\pi(\tilde{p}(0)) = a''$ and $\pi(\tilde{p}(1)) = a'$. Hence, $\pi(\tilde{p})$ is a path in S^n which connects a' and a''. This proves S_α^n is a connected set in R^n.

APPENDIX II

For a fixed n let $Q_{\alpha,\beta}^n = (q_{ij})$ be the matrix representation of $T_{\alpha,\beta}$ defined in section 3.1. Recall that for each fixed i, the row $(q_{i0}, q_{i1}, \ldots, q_{in})$ of $Q_{\alpha,\beta}$ represents the coefficients of

$$(s-\alpha)^{n-i}(s-\beta)^i = \sum_{j=0}^{n} q_{ij} s^{n-j} \qquad (II.1)$$

Now, let $f(x,y) = x^{n-i}y^i$. Then,

$$f(-\alpha+s, -\beta+s) = (s-\alpha)^{n-i}(s-\beta)^i \qquad (II.2)$$

In terms of Taylor's theorem, one has

$$f(a+h, b+k) = \frac{1}{(n-j)!} \sum_{j=0}^{n} (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^{n-j}(x^{n-i}y^i)\Big|_{\substack{x=a \\ y=b}}$$

By substituting $a = -\alpha$, $b = -\beta$ and $h = k = s$ into the above equation one obtains

$$(s-\alpha)^{n-i}(s-\beta)^i = \sum_{j=0}^{n} [\frac{1}{(n-j)!} (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})^{n-j}(x^{n-i}y^i)\Big|_{\substack{x=-\alpha \\ y=-\beta}}]s^{n-j} \qquad (II.3)$$

By comparing the right hand sides of (II.1) and (II.3), one derives another expression for the entries of $Q_{\alpha,\beta}$, namely,

$$q_{ij} = \frac{1}{(n-j)!} (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})^{n-j}(x^{n-i}y^i)\Big|_{\substack{x=-\alpha \\ y=-\beta}} \qquad (II.4)$$

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To show that for each fixed n, $\det(Q^n_{\alpha,\beta}) = (\alpha-\beta)^{n(n+1)/2}$, let $M^n(x,y) = (m^n_{i,j})$ be the $(n+1) \times (n+1)$ matrix with

$$m^n_{ij} = \frac{1}{(n-j)!} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{n-j} (x^{n-i}y^i)$$

We claim that $\det(M^n(x,y)) = (y-x)^{n(n+1)/2}$. To prove this, we use mathematical induction on n. Let $\tilde{M} = (\tilde{m}^n_{i,j})$ be the matrix obtained from $M^n(x,y)$ by starting from the $(n+1)$-th row of $M^n(x,y)$, subtracting successively the $(j-1)$-th row from the j-th row, $j = n+1, n, \ldots, 2$. These are elementary row operations which do not change the determinant of $M^n(x,y)$. Therefore, $\det(M^n(x,y)) = \det(\tilde{M})$. Furthermore, for any $i \geq 2$, $j \geq 2$,

$$\tilde{m}^n_{i,j} = \frac{1}{(n-j)!}\left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{n-j} (x^{n-i}y^i) - \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{n-j} (x^{n-i+1}y^{i-1}) \right]$$

$$= \frac{1}{(n-j)!}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{n-j}\left[(x^{n-i}y^{i-1})(y-x) \right]$$

$$= \frac{1}{(n-j)!}\left\{ (y-x)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{n-j}(x^{n-i}y^{i-1}) + (x^{n-i}y^{i-1}) \right.$$

$$\left. \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{n-j}(y-x) \right\}$$

$$= \frac{1}{(n-j)!}(y-x)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{n-j}(x^{n-i}y^{i-1})$$

$$= (y-x)m^{n-1}_{i-1,j-1}$$

Also, $\tilde{m}_{i1} = 0$ for $i \geq 2$, since $m^n_{i1} = 1$ for each $i = 1, 2, \ldots, n+1$. Hence, by factoring $(y-x)$ out of 2nd, 3rd, and $(n+1)$-th row of M and applying induction hypothesis, one obtains

$$\det(M^n(x,y)) = (y-x)^n \det(M^{n-1}(x,y)) = (y-x)^n \cdot (y-x)^{(n-1)n/2}$$

$$= (y-x)^{n(n+1)/2}$$

Now, observe that $M^n(-\alpha,-\beta) = Q^n_{\alpha,\beta}$. Therefore,

$$\det(Q^n_{\alpha,\beta}) = \det(M^n(x,y))\Big|_{\substack{x=-\alpha \\ y=-\beta}} = (\alpha-\beta)^{n(n+1)/2}$$

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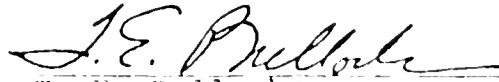
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